AUSTRALIAN MATHEMATICAL OLYMPIADS 1996–2011

BOOK 2

H LAUSCH, A DI PASQUALE, DC HUNT & PJ TAYLOR

AMT Publishing

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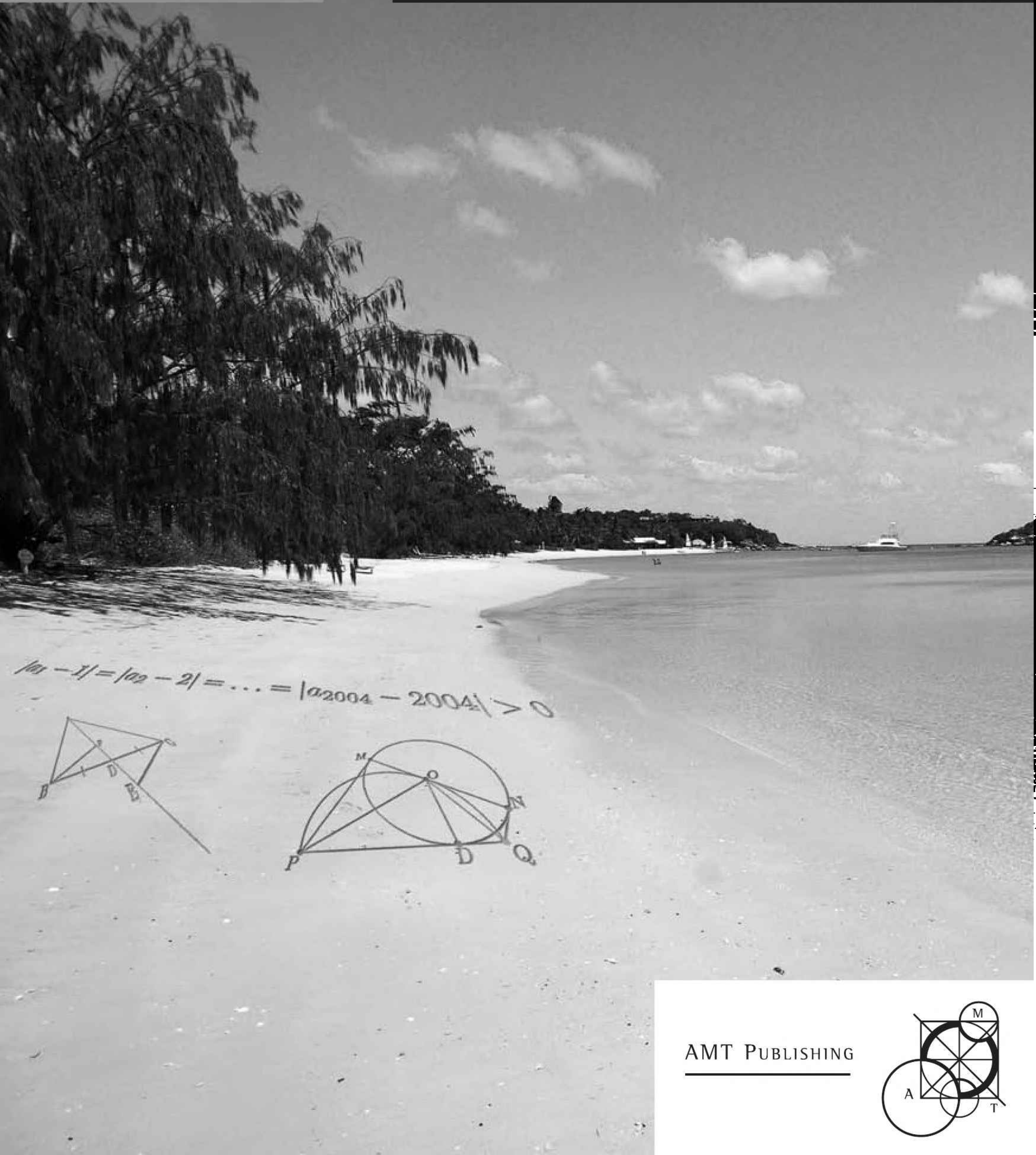

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PREFACE

The Australian Mathematical Olympiad Problems for the first years were published in a book with the same name but sub-headed "Book 1 1979 to 1995". The Australian Mathematical Olympiad has continued its traditions in the following years and we now have enough to publish a second book. During the time covered by this book, unlike the first book, there have only been one Problems Committee Chairman and two Team Leaders, and all three (Lausch, Hunt and Di Pasquale) have been available to work through our materials and in fact improve them somewhat in places. As a result we believe this book will be an accurate and useful resource for the student aspiring to develop their mathematical knowledge in a systematic manner.

Whereas the procedures have been refined and improved during the past 16 years, because the event was already developed, there is not so much history to tell. However we reproduce below the Preface from the recent reprint of Book 1 because it does give useful background information.

Preface of Book 1

Even though it did not formally come into existence until 1980 the history of the Australian Mathematical Olympiad Committee (AMOC) really began in the 1970s. A number of mathematicians in Australia had become aware of the growing strength of the International Mathematical Olympiad, which had commenced in 1959 in Romania with six Eastern Bloc countries taking part. The following extract comes from the first Annual Report of AMOC:

"The possibility that Australia might take part in the International Olympiad Programme had been under discussion informally in various places for some years when an invitation to the 1979 Olympiad held in London reached the Australian Government. Although this invitation had to be declined because no procedures were available for selecting a team and sending it to the Olympiad, its arrival did have the effect of stimulating action which led to the formation of the Australian Mathematical Olympiad Committee.

"Two particular consequences of the 1979 invitation should be mentioned. First, Mr JL Williams was able to arrange to attend the London Olympiad as an observer and, with the assistance particularly of the USA team under the leadership of Professor SL Greitzer, to observe the proceedings at the Olympiad. On his return Mr Williams prepared a very useful report which has been of great value to those involved in the Olympiad movement.

"Second, the Australian Mathematical Society established an interim Australian Mathematical Olympiad Committee under the Chairmanship of Mr PJ O'Halloran. This committee arranged for a pilot series of monthly practice sets of problems to be sent to interested students from whom a team might be chosen for the 1980 IMO and set up a panel of State Organisers to arrange for activities associated with the programme in each state and territory. In the event, the 1980 IMO, which was to be held in Outer Mongolia, did not take place."

It should be noted that the first Australian Mathematical Olympiad also took place during the life of the interim committee, in August 1979. The timing later changed, so the second Australian Mathematical Olympiad was not held until 1981 (April). Australia's first entry at the IMO then took place at Washington in 1981.

Whereas Peter O'Halloran and Jim Williams were the two most prominent figures in the establishment of AMOC, many others were involved and their names are acknowledged in the following section and the Honour Roll at the end of the book.

The format of the Australian Mathematical Olympiad paper has changed slightly over the years, and its date brought back from April to February to enable the Asian Pacific Mathematics Olympiad to be used also in selecting the Australian team at the IMO. Over the past few years participation has stabilised at about 100 students per year, generally those identified by State Directors.

The highlight for AMOC over the years was in 1988, when Australia hosted the IMO in Canberra. This was the first staging of an IMO in the southern hemisphere, and this enabled a number of new countries to participate for the first time.

The other major development for AMOC in recent years was the development in the 1990s of the Mathematics Challenge for Young Australians. This event has attracted over 13,000 students annually, from a Challenge stage, in which students are given three weeks to solve six challenging problems, to the Enrichment stage providing a choice of six courses of extension study. This event, under the leadership of Bruce Henry, has increased the quantity and quality of students participating in the Olympiad program, as well as providing higher mathematical experience to a wider number of Australian students.

The development of a program to participate at the IMO is a long and painstaking process. At the time of this book going to press, it is difficult to ignore the results of the just completed IMO in Argentina. At this IMO Australia has achieved by far its best placing, 9th out of 82 participating countries, with 2 gold medals, 3 silver medals and 1 bronze medal. A number of those who founded AMOC many years ago have either passed away (Peter O'Halloran and Jim Williams) or retired after a significant input to AMOC and were only able to dream of such a result. The result is, nevertheless, a tribute to those pioneers.

HL, A DiP, DCH and PJT
December 2011

ACKNOWLEDGEMENTS AND OBSERVATIONS

As much as is possible, the people who were responsible for developing the academic material for this book are acknowledged for their roles either as problem creators (Senior Problems Committee members), moderators and authors of solutions (many being students) or in the honour roll at the end. It is impossible to be certain that the names of all composers and solvers are present or accurate, but it is believed that these acknowledgements are close to accurate and complete.

There may be some apparent inconsistencies in the way in which awards were named and students' names announced. It was decided after some efforts at creating uniformity to announce exactly as they were announced at the time and recorded in AMOC's annual book *The Australian Scene.*

Generally results of AMOs have had the actual marks withheld, but the reason for not having alphabetic order within categories is that normally the names are in ranked order. An asterisk will denote a perfect score.

Peter Taylor
July 2011

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1996

Problems

PAPER 1 Tuesday 06 February

1. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.

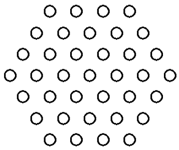
2. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that

$$\frac{p(\frac{1}{2}) + p(-\frac{1}{2})}{p(0)} = 1000.$$

Find the value of $\dfrac{1}{r_1 r_2} + \dfrac{1}{r_2 r_3} + \dfrac{1}{r_3 r_1}$.

3. Identical tubes are bundled together into a hexagonal form:

The number of tubes in the bundle can be 1, 7, 19, 37 (as shown), 61, 91, ... If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69.



What is the 69th number in the sequence which ends in 69?

4. For which positive integers n can we rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ in such a way that

$$|a_k - k| = |a_1 - 1| \neq 0$$

for $k = 2, 3, \ldots, n$?

PAPER 2 Wednesday 07 February

5. Let $a_1, a_2, \ldots, a_n$ be real numbers and s a non-negative real number such that

(i) $a_1 \leq a_2 \leq \ldots \leq a_n$;

(ii) $a_1 + a_2 + \ldots + a_n = 0$;

(iii) $|a_1| + |a_2| + \ldots + |a_n| = s$.

Prove that

$$a_n - a_1 \geq \frac{2s}{n}.$$

6. Let $ABCD$ be a cyclic quadrilateral and let P and Q be points on the sides AB and AD respectively such that $AP = CD$ and $AQ = BC$. Let M be the point of intersection of AC and PQ. Show that M is the midpoint of PQ.

7. For each positive integer n, let $\sigma(n)$ denote the sum of all positive integers that divide n. Let k be a positive integer and $n_1 < n_2, \ldots$ be an infinite sequence of positive integers with the property that $\sigma(n_i) - n_i = k$ for $i = 1, 2, \ldots$. Prove that n_i is a prime for $i = 1, 2, \ldots$.

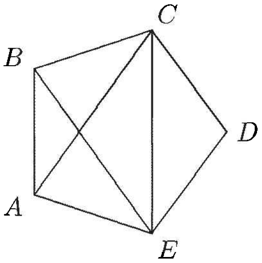
8. Let f be a function that is defined for all integers and takes only the values 0 and 1. Suppose f has the following properties:

(i) $f(n + 1996) = f(n)$ for all integers n;

(ii) $f(1) + f(2) + \cdots + f(1996) = 45$.

Prove that there exists an integer t such that $f(n + t) = 0$ for all n for which $f(n) = 1$.

Solution 1



We have

$$\begin{aligned}\angle BEC &= \angle ECD && \text{since } EB\|CD\\ &= \angle DEC && \text{since } ED = DC\\ &= \angle ECA && \text{since } AC\|ED\\ &= \angle CAB && \text{since } AB\|EC.\end{aligned}$$

Thus $\angle BEC = \angle BAC$ which means that $ABCE$ is a cyclic quadrilateral. Therefore $AE = BC$ since $\angle ECA = \angle BEC$.
Similarly we can show that $ED = AB$ so that all sides are equal.

Next, we can see that $\angle EDC = \angle DCB$ since these angles are the base angles of an isosceles trapezium. Similarly, $\angle DCB = \angle CBA$ and so on.

Therefore $ABCDE$ is a regular pentagon.

Solution 2

Let $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.
Then $r_1 + r_2 + r_3 = -\frac{a_2}{a_3}$ and $r_1r_2r_3 = -\frac{a_0}{a_3}$. Hence

$$\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1} = \frac{r_1+r_2+r_3}{r_1r_2r_3} = \frac{a_2}{a_0}.$$

But

$$p\left(\frac{1}{2}\right) = \frac{a_3}{8} + \frac{a_2}{4} + \frac{a_1}{2} + a_0$$

and

$$p\left(-\frac{1}{2}\right) = -\frac{a_3}{8} + \frac{a_2}{4} - \frac{a_1}{2} + a_0.$$

Therefore

$$1000 = \frac{p(\frac{1}{2}) + p(-\frac{1}{2})}{p(0)} = \frac{\frac{a_2}{2} + 2a_0}{a_0} = \frac{a_2}{2a_0} + 2.$$

Hence

$$\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1} = \frac{a_2}{a_0} = 2(1000-2) = 1996.$$

Solution 3

Alternative 1

Let t_n be the nth term of the sequence with $n = 1, 2, \ldots$.
Then we can see that $t_1 = 1$ and $t_{n+1} = t_n + 6n$ for $n = 1, 2, \ldots$. Hence

$$\begin{aligned} t_n &= 1 + 6(1 + 2 + \cdots + (n-1)) \\ &= 1 + 6 \times \frac{n(n-1)}{2} \\ &= 3n^2 - 3n + 1. \end{aligned}$$

We first want to find all those values of n for which t_n is a number ending in 69. Algebraically it means that $3n^2 - 3n + 1 - 69$ is divisible by 100. Hence $3n^2 - 3n + 1 - 169$ is divisible by 100 which implies that $n^2 - n - 56$ is divisible by 100 since 3 and 100 are relatively prime.
Thus we have to solve the quadratic equation

$$n^2 - n - 56 = 100t,$$

where t is an integer. It has the solutions

$$n = \frac{1}{2}(1 \pm 5\sqrt{9 + 16t})$$

and so

$$9 + 16t = N^2$$

where N is an integer.

Hence $N^2 - 9$ is divisible by 16. Clearly N is an odd number so there is an integer K such that $N = 2K + 1$. Hence $(2K + 1)^2 - 9$ is divisible by 16 which means that $K^2 + K - 2$ is divisible by 4.

Since $K^2 + K - 2 = (K - 1)(K + 2)$, we see that either $K - 1$ or $K + 2$ is divisible by 4. This implies that $K = 4M + 1$ or $K = 4M + 2$ where M is an integer.

Therefore $N = 8M + 3$ or $N = 8M + 5$.
Hence $n = 20M + 8$ or $n = 20M + 13$ where $M = 0, 1, 2, 3, \ldots$.
Therefore the 69th member of the sequence $20M + 8$, $20M + 13$ where $M = 0, 1, 2, 3, \ldots$ is $20 \times 34 + 8 = 688$, so that

$$t_{688} = 1\,417\,969$$

is the required number.

Alternative 2 (David Hunt)

Let t_n be the nth term of the sequence with $n = 1, 2, \ldots$.

$$t_1 = 6 \quad \text{and} \quad t_{n+1} = t_n + 6n \quad \text{for } n = 1, 2, \ldots$$

n	1	2	3	4	5	6	7	8	9	10	11	...
$6n$	6	12	18	24	30	36	42	48	54	60	66	
t_n	1	7	19	37	61	91	127	169	217	271	331	

We see that n must be $\equiv 3 \pmod 5$.

n	3	8	13	18	23	28	33	...
t_n	19	169	469	919	1519	1669	2569	

The differences are 150, 300, 450, 600, ...

So t_n ends in 69 iff $n \equiv 8$ or 13 mod 20. The 69th such number is t_n where $n = 8 + 20 \times 34 = 688$ with

$$t_{688} = 1\,417\,969.$$

Formal proofs of the above facts are obvious.

Solution 4

We claim that this is possible if and only if n is even.

Let n be even, say $n = 2m$ for some positive integer m.
Take $(a_1, a_2, \ldots, a_n) = (m+1, m+2, \ldots, 2m, 1, 2, \ldots, m)$. Then it is easily seen that $|a_k - k| = m$ for every $k = 1, 2, \ldots, n$. Alternatively, take $(a_1, a_2, \ldots, a_n) = (2, 1, 4, 3, \ldots, n, n-1)$ with $|a_k - k| = 1$ for all k.

Now let n be odd, and suppose that we can rearrange the sequence $1, 2, \ldots, n$ to $a_1, a_2, \ldots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \ldots, n$. Let G be the number of ordered pairs (a_k, k) with $a_k > k$ and L the number of pairs (a_k, k) with $a_k < k$. Then $G + L = n$. Also

$$(a_1-1)+(a_2-2)+\cdots+(a_n-n) = (a_1+a_2+\cdots+a_n)-(1+2+\cdots+n) = 0.$$

Let $|a_k - k| = d$, then $a_k - k = d$ if $a_k > k$ and $a_k - k = -d$ if $a_k < k$. Hence

$$(a_1 - 1) + (a_2 - 2) + \cdots + (a_n - n) = Gd - Ld = 0,$$

which means $G - L = 0$, that is, $G = L$.
Hence $n = 2G$, which is a contradiction as we assumed that n is odd.

Solution 5

Alternative 1

By (i) and (ii), we have $a_1 \leq \ldots \leq a_r \leq 0 \leq a_{r+1} \leq \ldots \leq a_n$ for some integer r such that $1 \leq r < n$.

Let $m = \dfrac{a_1 + \cdots + a_r}{r}$ and $M = \dfrac{a_{r+1} + \cdots + a_n}{n-r}$, then

$$rm + (n-r)M = a_1 + a_2 + \cdots + a_n = 0$$

and

$$-rm + (n-r)M = |a_1| + |a_2| + \cdots + |a_n| = s.$$

Thus we have $rm + (n-r)M = 0$ and $-rm + (n-r)M = s$ which implies that $m = -\frac{s}{2r}$ and $M = \frac{s}{2(n-r)}$.

Since $a_1 \leq \cdots \leq a_r$, we have

$$m = \frac{a_1 + \cdots + a_r}{r} \geq \frac{a_1 r}{r} = a_1.$$

Hence $m \geq a_1$. Similarly $M \leq a_n$.
Therefore

$$\begin{aligned} a_n - a_1 &\geq M - m \\ &= \frac{s}{2}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\ &= \frac{s}{2}\frac{n}{(n-r)r} \\ &= \frac{2sn}{n^2 - (n-2r)^2} \\ &\geq \frac{2sn}{n^2} \\ &= \frac{2s}{n}. \end{aligned}$$

Thus $a_n - a_1 \geq \frac{2s}{n}$.

Alternative 2 (David Hunt)

By (i) and (ii), we have $a_1 \leq \cdots \leq a_r \leq 0 \leq a_{r+1} \leq \cdots \leq a_n$ for some r with $1 \leq r < n$.

If we replace $a_1, \ldots, a_r$ by their average b and replace $a_{r+1}, \ldots, a_n$ by their average a then $a_n - a_1 = a - b$ is reduced. So we may assume there are r bs with $b < 0$ and $n - r$ as with $a > 0$.

By (ii),

$$rb + (n-r)a = 0. \qquad (1)$$

By (iii)

$$-rb + (n-r)a = s. \qquad (2)$$

Solving (1) and (2)

$$a = \frac{s}{2(n-r)} \quad \text{and} \quad b = -\frac{s}{r}.$$

Hence

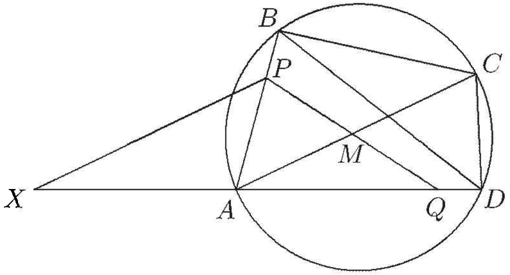
$$\begin{aligned} a_n - a_1 &= a - b \\ &= \frac{s}{2}\left(\frac{1}{n-r} + \frac{1}{r}\right) \\ &= \frac{s}{2}\,\frac{n}{(n-r)r} \\ &= \frac{2sn}{n^2 - (n-2r)^2} \\ &\geq \frac{2sn}{n^2} \\ &= \frac{2s}{n}. \end{aligned}$$

Note that equality can only occur when n is even.

Solution 6

Alternative 1

Let X be a point on the straight line DA such that $XA = DA$.



Then $/PAX = 180° - /PAQ$.
But $/BAD + /BCD = 180°$ since the quadrilateral $ABCD$ is cyclic.
Hence

$$/PAX = 180° - /PAQ = 180° - /BAD = /BCD.$$

Thus $/PAX = /DCB$.
Furthermore, $AX = AQ = CB$ and $AP = CD$.
Therefore the triangles XAP and BCD are congruent.
Hence $/AXP = /CBD$.
But $/CBD = /CAD$ as they are subtended by the same arc.
Hence $/PXA = /CAD$ which implies that XP is parallel to AM.
Since $XA = AQ$ and AM is parallel to XP, it follows that $PM = MQ$.

Alternative 2

Let $AP = p$, $AQ = q$. Then $BC = q$ and $CD = p$.
Let also $\angle BAC = \angle BDC = \alpha$, $\angle DAC = \angle DBC = \beta$ and $\angle PMA = \gamma$.
By the sine rule in $\triangle PAM$, $\dfrac{PM}{\sin\alpha} = \dfrac{p}{\sin\gamma}$, hence $PM = \dfrac{p\sin\alpha}{\sin\gamma}$.
By the sine rule in $\triangle QAM$, $\dfrac{QM}{\sin\beta} = \dfrac{q}{\sin(180^\circ - \gamma)}$, hence $QM = \dfrac{q\sin\beta}{\sin\gamma}$.
Also by the sine rule in $\triangle BCD$, $\dfrac{q}{\sin\alpha} = \dfrac{p}{\sin\beta}$, hence $p\sin\alpha = q\sin\beta$.
Therefore

$$PM = \frac{p\sin\alpha}{\sin\gamma} = \frac{q\sin\beta}{\sin\gamma} = QM.$$

Thus $PM = QM$, as required.

Solution 7

If n is a prime, then

$$\sigma(n) - n = n + 1 - n = 1.$$

If n is a composite, then n has a divisor with $\sqrt{n} \le d < n$, so that

$$\sigma(n) - n \ge n + \sqrt{n} + 1 - n > \sqrt{n} > 1.$$

That is, we have shown that if $k = 1$, each n_i must be a prime.

If $k > 1$, all n_i are composite and $k = \sigma(n_i) - n_i > \sqrt{n_i}$ for $i = 1, 2, \ldots$. Thus $k > \sqrt{n_i}$ for $i = 1, 2, \ldots$, which is impossible for a strictly increasing sequence of positive integers.

We have shown $k = 1$ and that each n_i is prime.

Solution 8

Condition (ii) means that $f(n) = 1$ for precisely 45 of the 1996 integers $0, 1, 2, \ldots, 1995$. For all pairs of integers p and q such that $0 \le p \le q \le 1995$ and $f(p) = f(q) = 1$ we calculate the numbers $q - p$ and $1996 - q + p$. Since p and q are from a set of 45 numbers, there are at most $45 \times 44 + 1 = 1981$ possible distinct numbers $q - p$ and $1996 - q + p$. Hence there exists a number t such that $0 \le t \le 1995$ and t is equal to none of the numbers $q - p$ and $1996 - q + p$ for $0 \le p \le q \le 1995$. We shall prove that $f(n + t) = 0$ for all n for which $f(n) = 1$ holds.

Assume that there exists n such that $f(n + t) = 1$ and $f(n) = 1$. In view of (i), we can also assume that $0 \le n \le 1995$. Then either $n + t \le 1995$ or $1996 \le n + t \le 1996 + 1995$.
If $n + t \le 1995$ then we have $f(n + t) = 1$, $f(n) = 1$ and $t = (n + t) - n$ which contradicts the requirements imposed on t.

If $1996 \leq n+t \leq 1996+1995$ then $0 \leq n+t-1996 \leq 1995$ and we have $f(n)=1$, $f(n+t-1996)=1$ and $t=1996-n+(n+t-1996)$ which again is not possible.

Thus $f(n+t)=0$ for all n such that $f(n)=1$, as required.

RESULTS

Number of Candidates: 111

Excellence

Brian Scerri	Canberra Grammar School	ACT	12
Justin Ghan	Pembroke School	SA	10
Jian He	University High School	VIC	12
Alexandre Mah	North Sydney Boys High School	NSW	12
Daniel Mathews	Scotch College	VIC	11
Brett Parker	Penleigh and Essendon Grammar School	VIC	12
Gordon Ng	Sydney Grammar School	NSW	12
Andrew Yao	Gosford High School	NSW	12
Andrew Brooke-Taylor	Kirwan State High School	QLD	12
Daniel Ford	James Sheahan High School	NSW	12

Merit

Chang Chen	Newcastle Grammar School	NSW	12
Stephen Farrar	James Ruse Agricultural High School	NSW	10
Daniel Stitt	Sydney Grammar School	NSW	12
Jurn Ho	James Ruse Agricultural High School	NSW	12
Jonathon Kusilek	Hurlstone Agricultural High School	NSW	11
John Dethridge	Melbourne Grammar School	VIC	12
Martin Jenkins	Home studies	NSW	12
Kuhn Ip	Melbourne Grammar School	VIC	12
Paul Hunter	The Friends' School	TAS	12
Scott Morrison	Sydney Grammar School	NSW	11
Thomas Lam	Sydney Grammar School	NSW	11
David Philp	Caringbah High School	NSW	12
Norman Do	Melbourne Grammar School	VIC	11
Peter Andrewartha	Christ Church Grammar School	WA	11
Ben Cusack	Stirling College	ACT	12
James Henstridge	Hollywood Senior High School	WA	12
Adrian Wong	James Ruse Agricultural High School	NSW	12
Geoffrey Kong	Scotch College	VIC	11
Michael Lim	Sydney Grammar School	NSW	12

Australian Team to participate in 1996 IMO at Mumbai, India

Jian He, Alexandre Mah, Daniel Mathews, Brett Parker, Daniel Ford, John Dethridge, with Brian Scerri as reserve.

1997

Problems

PAPER 1 Tuesday 05 February

1. Let ABC be a triangle with $AB = AC$ and $\angle BAC < 120°$. Let D be the midpoint of BC. E is the point on AD such that $\angle AEB = 120°$. Let E' be any point on AD distinct from E. Prove that

$$EA + EB + EC < E'A + E'B + E'C.$$

2. Let $a_1, a_2, \ldots, a_k$ be real numbers satisfying the following two conditions:

(i) $0 \le a_1 \le a_2 \le \ldots \le a_k$;
(ii) $a_1 + a_2 + \cdots + a_k = 1$.

Prove that $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \le \dfrac{1}{k}$ for $n = 1, 2, \ldots, k$.

3. Determine all functions f defined for all real numbers and taking real values that satisfy the inequality

$$|f(x+h) - f(x)| \le h^2$$

for all real numbers x and h.

4. An infinite staircase is a sequence of ordered pairs of non-negative integers

$$(x_1, y_1), (x_1, y_2), (x_2, y_2), (x_2, y_3), (x_3, y_3), (x_3, y_4), (x_4, y_4), \ldots$$

in which $0 \le x_1 < x_2 < x_3 < \ldots$ and $0 \le y_1 < y_2 < y_3 < \ldots$.

Prove that if each point (x, y) in the coordinate plane, with x and y being non-negative integers, is coloured either red or blue, then it is possible to find an infinite staircase such that all its points are the same colour.

PAPER 2 Wednesday 06 February

5. For each positive integer n let $p(n)$ be the product of all positive integers that divide n. Prove that if a and b are positive integers and $p(a) = p(b)$, then $a = b$.

6. For any real number x, let $[x]$ denote the largest integer not exceeding x.

If n is a positive integer, prove that

$$[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}].$$

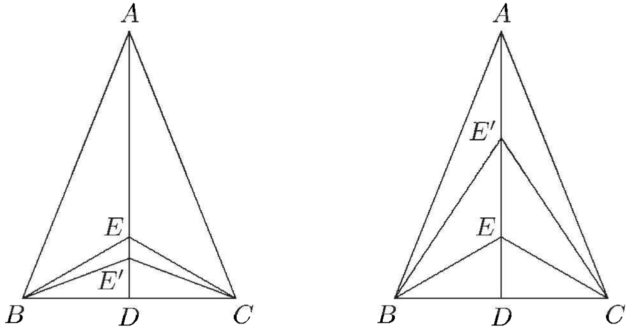
7. Let m and n be integers greater than 1. Prove that

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > 1.$$

8. Let ABC be a triangle with $\angle ABC = 60^\circ$ and $\angle BAC = 40^\circ$. Let P be the point on AB such that $\angle BCP = 70^\circ$ and let Q be the point on AC such that $\angle CBQ = 40^\circ$. Let BQ intersect CP at R.

Prove that AR (extended) is perpendicular to BC.

Solution 1 (Neil Williams)



Case 1. E' is on ED.
If

$$EB < \frac{1}{2}E'E + E'B$$

and

$$EC < \frac{1}{2}E'E + E'C$$

then

$$\begin{aligned} EA + EB + EC &< (EA + E'E) + E'B + E'C \\ &= E'A + E'B + E'C. \end{aligned}$$

By symmetry we only need to prove

$$EB < \frac{1}{2}E'E + E'B$$

or (since both sides are positive)

$$EB^2 < \frac{1}{4}E'E^2 + E'B^2 + E'E \times E'B. \qquad (1)$$

By the cosine rule applied to the triangle EBE'

$$E'B^2 = E'E^2 + EB^2 - 2E'E \times EB \times \cos 60^\circ = E'E^2 + EB^2 - E'E \times EB$$

so

$$EB^2 = -E'E^2 + E'B^2 + E'E \times EB < \frac{1}{4}E'E^2 + E'B^2 + E'E \times E'B,$$

as required for (1).

Case 2. E' is on EA.
As above, by symmetry the result is equivalent to

$$EB + \frac{1}{2}E'E < E'B$$

or (since both sides are positive)

$$EB^2 + \frac{1}{4}E'E^2 + E'E \times EB < E'B^2. \tag{2}$$

By the cosine rule applied to the triangle EBE'

$$\begin{aligned} E'B^2 &= E'E^2 + EB^2 - 2E'E \times EB \times \cos 120^\circ \\ &= E'E^2 + EB^2 + E'E \times EB \\ &> EB^2 + \frac{1}{4}E'E^2 + E'E \times EB \end{aligned}$$

as required for (2).

Solution 2

Alternative 1 (Neil Williams)

Observe that for each i with $1 \le i < k$ we have $a_1 + \cdots + a_i \le i a_{i+1}$, so

$$(i+1)(a_1 + \cdots + a_i) \le i(a_1 + \cdots + a_{i+1}).$$

Hence

$$\frac{a_1 + \cdots + a_i}{i} \le \frac{a_1 + \cdots + a_{i+1}}{i+1}.$$

Therefore

$$\frac{a_1}{1} \le \frac{a_1 + a_2}{2} \le \ldots \le \frac{a_1 + \cdots + a_n}{n} \le \ldots \le \frac{a_1 + \cdots + a_k}{k} = \frac{1}{k}.$$

Alternative 2 (David Hunt)

Suppose $\frac{a_1 + \cdots + a_n}{n} > \frac{1}{k}$. Then $a_1 + \cdots + a_n > \frac{n}{k}$. Hence $a_n > \frac{1}{k}$ and thus $a_{n+1}, \ldots, a_k > \frac{1}{k}$. Therefore

$$1 = a_1 + \cdots + a_k > \frac{n}{k} + (k-n)\frac{1}{k} = 1,$$

a contradiction.

Solution 3

Let n be a positive integer and x any real number. Then

$$\begin{aligned} f(0) - f(x) &= f(0) - f\left(\frac{x}{n}\right) + f\left(\frac{x}{n}\right) - f\left(\frac{2x}{n}\right) \\ &\quad + \ldots + f\left(\frac{(n-1)x}{n}\right) - f\left(\frac{nx}{n}\right). \end{aligned}$$

By the triangle inequality,

$$\begin{aligned}|f(0)-f(x)| &\le \left|f(0)-f\left(\frac{x}{n}\right)\right|+\ldots+\left|f\left(\frac{(n-1)x}{n}\right)-f\left(\frac{nx}{n}\right)\right| \\ &\le \left(\frac{x}{n}\right)^2+\left(\frac{x}{n}\right)^2+\cdots+\left(\frac{x}{n}\right)^2 \\ &= n\left(\frac{x}{n}\right)^2 \\ &= \frac{x^2}{n}.\end{aligned}$$

As this is true for all positive integers n, we must have $|f(0)-f(x)|=0$. Thus $f(x)=f(0)$ for all real numbers x. Every solution to the problem is therefore a constant function.

Conversely, let $f(x)=c$ for all real numbers x. Then $|f(x+h)-f(x)| = |c-c| = 0 \le h^2$.

Therefore the set of all constant functions is the required set of functions.

Solution 4

If there are infinitely many horizontal lines each of which contains only a finite number of blue points, then we can form a staircase as follows. Suppose that those horizontal lines are $y=y_1$, $y=y_2$, $y=y_3,\ldots$. Take a sufficiently large positive integer x_1 such that for all $x \ge x_1$ the points (x,y_1) and (x,y_2) are red; then take x_2 such that for all $x \ge x_2$ the point (x,y_3) is red and so on. Our staircase is then (x_1,y_1), (x_1,y_2), (x_2,y_2), (x_2,y_3), $(x_3,y_3),\ldots$.

The same argument applies if there are infinitely many horizontal lines each of which contains only a finite number of red points, or if there are infinitely many vertical lines which contain only a finite number of points of one colour.

If none of these conditions applies, then there is a point (x_1,y_1) such that all vertical lines $x=a$ and all horizontal lines $y=b$ for $a \ge x_1$ and $b \ge y_1$ contain infinitely many points of either colour. We can then form a staircase by starting with (x_1,y_1), then moving up till we find a point of the same colour, then till we find another of the same colour, and so on.

Solution 5

For each positive integer n, let $d(n)$ be the number of positive integers that divide n. Then $p(n)=n^{\frac{d(n)}{2}}$. We have thus to show that

$$a^{\frac{d(a)}{2}} = b^{\frac{d(b)}{2}} \tag{1}$$

implies $a=b$.

Clearly (1) implies that a and b have the same prime factors.
Let $a = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ and $b = p_1^{\beta_1} \cdots p_r^{\beta_r}$ be the prime decompositions of a and b where α_i, β_i are positive for $i = 1, \ldots, r$. Then by (1), $\alpha_i d(a) = \beta_i d(b)$ for $i = 1, \ldots, r$. Hence $\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \ldots = \frac{\alpha_2}{\beta_2}$. If this ratio equals 1, we have $a = b$.

Otherwise we may assume without loss of generality that this ratio is greater than 1. But then $d(b) < d(a)$, which would imply that $p(b) < p(a)$, a contradiction.

Solution 6

We have

$$\begin{aligned}(\sqrt{n} + \sqrt{n+1})^2 &= n + 2\sqrt{n^2+n} + n + 1 \\ &= 2n + 1 + 2\sqrt{n^2+n}.\end{aligned}$$

Since $n^2 < n^2 + n < n^2 + 2n + 1$, we have $n < \sqrt{n^2+n} < n + 1$.
Hence $4n + 1 < (\sqrt{n} + \sqrt{n+1})^2 < 4n + 3$.
It follows that $[\sqrt{4n+1}] < [\sqrt{n} + \sqrt{n+1}] < [\sqrt{4n+3}]$.

Note that neither $4n + 2$ nor $4n + 3$ can be a perfect square since the square of an integer can only have remainders 0 and 1 when divided by 4.

Hence $[\sqrt{4n+1}] = [\sqrt{4n+3}]$ and therefore $[\sqrt{n} + \sqrt{n+1}] = [\sqrt{4n+1}]$.

Solution 7

By the binomial theorem, $(1+x)^m > 1+mx$ for all positive real numbers x.
Hence $\left(1 + \frac{n}{m}\right)^m > 1 + n$. Therefore

$$\sqrt[m]{n+1} < 1 + \frac{n}{m}. \tag{1}$$

Similarly,

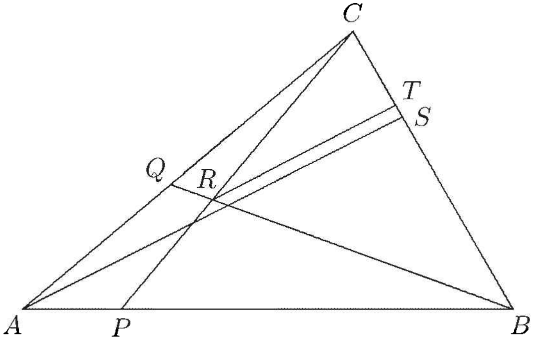
$$\sqrt[n]{m+1} < 1 + \frac{m}{n}. \tag{2}$$

By (1) and (2),

$$\frac{1}{\sqrt[m]{n+1}} + \frac{1}{\sqrt[n]{m+1}} > \frac{m}{m+n} + \frac{n}{m+n} = 1$$

as required.

Solution 8



Let the perpendiculars from A and R to BC meet BC at S and T respectively.
Then $BS = AB\cos 60^\circ$.

Also $\angle ACB = 180^\circ - 60^\circ - 40^\circ = 80^\circ$.
By the sine rule,

$$\frac{AB}{\sin 80^\circ} = \frac{BC}{\sin 40^\circ}.$$

Hence

$$AB = \frac{BC\sin 80^\circ}{\sin 40^\circ} = \frac{2BC\sin 40^\circ\cos 40^\circ}{\sin 40^\circ} = 2BC\cos 40^\circ.$$

It follows that $BS = 2BC\cos 40^\circ\cos 60^\circ = BC\cos 40^\circ$.
In the triangle BCR, $\angle BRC = 180^\circ - 40^\circ - 70^\circ = 70^\circ$. Hence BCR is an isosceles triangle with $BC = BR$. Hence $BS = BR\cos 40^\circ$.

On the other hand, $BT = BR\cos 40^\circ$. Hence $BS = BT$.

Therefore S and T coincide which means that AR (extended) is perpendicular to BC.

RESULTS

Number of Candidates: 87

Excellence

Thomas Lam	Sydney Grammar Scool	NSW	12
Justin Ghan	Pembroke School	SA	11
Daniel Mathews	Scotch College	VIC	12
Damian Bramanis	Applecross Senior High School	WA	12
Jonathon Kusilek	Hurlstone Agricultural High School	NSW	12
Norman Do	Melbourne Grammar School	VIC	12
John Dethridge	Melbourne Grammar School	VIC	12
James Yun	John Paul College	QLD	12
Stephen Farrar	James Ruse Agricultural High School	NSW	11

Merit

Andrew Cheeseman	Mentone Grammar School	VIC	10
Chang Chen	Newcastle Grammar School	NSW	12
Hiroshi Miyazaki	Scotch College	VIC	11
Adam Solomon	Sydney Boys High School	NSW	11
Ben Toner	Melbourne Grammar School	VIC	12
David Varodayan	Sydney Grammar School	NSW	11
Shaun Chou	James Ruse Agricultural High School	NSW	12
Steven Irrgang	Carringbah High School	NSW	11
Geoffrey Kong	Scotch College	VIC	12
Martin Jenkins	Home Studies	NSW	12
Adrian Ratnapala	Kenmore State High School	QLD	12
Daniel Trambaiolo	Sydney Grammar School	NSW	12
Karen Ho	Queenwood School	NSW	11
Scott Morrison	Sydney Grammar School	NSW	12
Paul Childs	Sydney Technical High School	NSW	12
David Sekers	Moriah College	NSW	12
Peter Andrewartha	Christ Church Grammar School	WA	12
Will Moase	The Hutchins School	TAS	11

Australian Team to participate in 1997 IMO at Mar del Plata, Argentina

Thomas Lam, Justin Ghan, Daniel Mathews, Jonathan Kusilek, Norman Do, Stephen Farrar.

1998

Problems

PAPER 1 Wednesday 11 February

1. Determine all real solutions of the equation

$$(x+1998)(x+1999)(x+2000)(x+2001)+1=0.$$

2. Find all pairs (r, s) of non-negative real numbers that satisfy the following two conditions:

(i) $2^{r^4+s^2}+2^{r^2+s^4}=8$;
(ii) $r+s=2$.

3. Let ABC be a triangle, let D be a point on AB and E be a point on AC such that DE and BC are parallel and DE is a tangent to the incircle of the triangle ABC.

Prove that

$$8DE \leq AB+BC+CA.$$

4. Determine all functions f with real values that satisfy the following conditions:

(i) f is defined for all non-zero real numbers;
(ii) $f(-x)=f(x)$ for all non-zero real numbers x;
(iii) $f\left(\frac{1}{x+y}\right)=f\left(\frac{1}{x}\right)+f\left(\frac{1}{y}\right)+2(xy-1000)$ for all non-zero real numbers x and y such that $x+y \neq 0$.

PAPER 2 Thursday 12 February

5. Consider all 2×2 arrays with each entry being either 0 or 1. We say that a pair of such 2×2 arrays A and B is compatible if there exists a 3×3 array within which both A and B appear as 2×2 arrays. For example, the pair

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is compatible since both arrays occur within $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Determine all pairs of 2×2 arrays that are not compatible.

6. Prove that for any positive integer n,

$$(1998n)! \leq \left(\frac{3995n+1}{2} \cdot \frac{3993n+1}{2} \cdot \frac{3991n+1}{2} \cdot \cdots \cdot \frac{n+1}{2} \right)^n .$$

7. Let ABC be a triangle whose area is 1998 cm^2. Let G be the centroid of the triangle ABC. Each line through G cuts the triangle ABC into two regions having areas A_1 and A_2, say. Determine (with proof) the largest possible value of $A_1 - A_2$.

8. A team of archaeologists was able to establish that forty thousand years ago there had been a flourishing civilisation in the crocodile infested jungle of Udakak. As many as 12 cities were excavated. After some effort, the Udakak script was deciphered, and the rich Udakak literature was studied by historians.

They discovered that:

(i) Udakak could not have had more than 28 cities;

(ii) the Udakakan cities had peculiar trade arrangements:

  (a) whenever a city A had no trade relations with another city B, there were exactly two cities with which both A and B had trade relations;

  (b) whenever a city A had trade relations with another city B, no city had trade relations with both A and B.

For budgetary reasons, the archaeologists wanted to know how much excavation work was still lying ahead of them. Show how the archaeologists can work out the exact number of Udakakan cities.

Solution 1

Put $y = x + \frac{3999}{2}$. Then the equation becomes

$$\left(y-\frac{3}{2}\right)\left(y-\frac{1}{2}\right)\left(y+\frac{1}{2}\right)\left(y+\frac{3}{2}\right)+1=0.$$

Therefore $\left(y^2-\frac{9}{4}\right)\left(y^2-\frac{1}{4}\right)+1=0$ which simplifies to

$$\left(y^2-\frac{5}{4}\right)^2=0.$$

Hence $y = \pm\frac{\sqrt{5}}{2}$ and $x = \frac{\pm\sqrt{5}-3999}{2}$.

Solution 2

It is easy to see that $(r, s) = (1, 1)$ satisfies both (i) and (ii).
Condition (ii) implies that $0 \le r \le 2$ and $0 \le s \le 2$.

Put $r = 1 + t$. Then $-1 \le t \le 1$ and $s = 1 - t$. Hence condition (i) becomes

$$2^{(1+t)^4+(1-t)^2}+2^{(1+t)^2+(1-t)^4}=8,$$

or

$$2^{2+2t+7t^2+4t^3+t^4}+2^{2-2t+7t^2-4t^3+t^4}=2^3.$$

Therefore

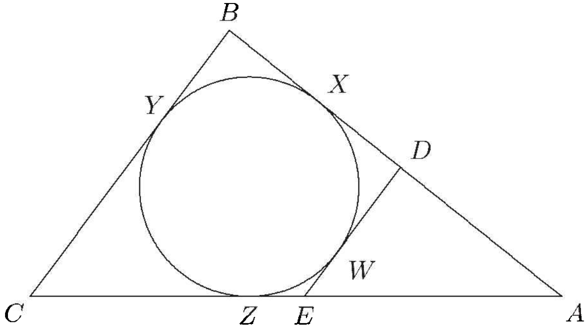
$$2^{2t+4t^3}+2^{-2t-4t^3}=2^{1-7t^2-t^4}.$$

Since the left-hand side of this equation is of the form $a+\frac{1}{a}$ with $a > 0$, it must be at least 2. On the other hand, the right-hand side is, clearly, at most 2. It follows that $t = 0$.

Thus $(1, 1)$ is the only solution.

Solution 3

Alternative 1



Let the points of contact of the incircle of triangle ABC with AB, BC, CA and DE be X, Y, Z and W respectively. Then

$$\begin{aligned}
&AD+DE+EA+2BC\\
&\qquad = AD+DW+WE+EA+2BY+2YC\\
&\qquad = AD+DX+ZE+EA+XB+BY+YC+CZ\\
&\qquad = AX+XB+AZ+CZ+BY+YC\\
&\qquad = AB+CA+BC.
\end{aligned}$$

Since DE and BC are parallel, triangles ADE and ABC are similar. Hence

$$\frac{DE}{BC}=\frac{AD+DE+EA}{AB+CA+BC}.$$

Therefore

$$\begin{aligned}
&\frac{DE}{AB+CA+BC}\\
&\qquad = \frac{BC(AD+DE+EA)}{(AB+CA+BC)^2}\\
&\qquad = \frac{BC(AD+DE+EA)}{(AD+DE+EA+2BC)^2}\\
&\qquad = \frac{BC(AD+DE+EA)}{(AD+DE+EA-2BC)^2+8BC(AD+DE+EA)}\\
&\qquad \le \frac{BC(AD+DE+EA)}{8BC(AD+DE+EA)}\\
&\qquad = \frac{1}{8}.
\end{aligned}$$

Alternative 2 (Angelo Di Pasquale)

We use the incircle substitution. Refer to the diagram in the first alternative. Let $x = BY = BX$, $y = CY = CZ$ and $z = AX = AZ$. Let $m = DX = DW$ and $n = EW = EZ$. Thus $AD = z - m$ and $AE = z - n$. Then from similar triangles ABC and ADE we have

$$\frac{z-m}{z+x} = \frac{z-n}{z+y} = \frac{m+n}{x+y} = r,$$

where r is the ratio of similtude. Using addendo we also find upon adding the three numerators and three denominators together that

$$r = \frac{2z}{2x+2y+2z} = \frac{z}{z+y+z}.$$

Thus

$$\frac{m+n}{x+y} = \frac{z}{x+y+z}.$$

Since $AB + BC + CA = 2(x+y+z)$, the inequality to be proven is equivalent to

$$\frac{8z(x+y)}{x+y+z} \leq 2(x+y+z).$$

Cross multiplying and simplifying shows that this is equivalent to proving that

$$x^2 + y^2 + z^2 + 2xy - 2xz - 2yz \geq 0.$$

But the left-hand side is equal to $(x+y-z)^2$ and is thus non-negative, as required.

Solution 4

Put $h(x) = f\left(\frac{1}{x}\right) - x^2 - 2000$. Then by (i), $h(x)$ is defined for all non-zero real numbers x; by (ii), $h(-x) = h(x)$ for all non-zero real numbers; and by (iii), $h(x+y) = h(x) + h(y)$ for all non-zero real numbers x and y with $x + y \neq 0$. Now let $x \neq 0$, $y \neq 0$ and $x + y \neq 0$. Then

$$\begin{aligned} h(x) &= h(x+y-y) \\ &= h(x+y) + h(-y) \\ &= h(x+y) + h(y) \\ &= h(x) + 2h(y). \end{aligned}$$

Hence $h(y) = 0$ for all $y \neq 0$.

It follows that $f\left(\frac{1}{x}\right) = x^2 + 2000$ for all $x \neq 0$.

One easily verifies that $f(x) = \dfrac{1}{x^2} + 2000$ for all $x \neq 0$ satisfies conditions (i), (ii) and (iii).

Thus $f(x) = \dfrac{1}{x^2} + 2000$ for all $x \neq 0$.

Solution 5

We shall show that the only such pair is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ is a pair that is not compatible. Then, if $d = x$, the pair would be compatible since both arrays occur within $\begin{pmatrix} a & b & * \\ c & d & y \\ * & z & w \end{pmatrix}$ where each $*$ is either 0 or 1.

We conclude that $d \neq x$, that is, $x = 1 - d$. Similarly we show that $y = 1 - c$, $z = 1 - b$ and $w = 1 - a$.

Also $b \neq x$ or $d \neq z$ as otherwise both arrays would occur within an array of the form $\begin{pmatrix} a & b & y \\ c & d & w \\ * & * & * \end{pmatrix}$. Hence $x = 1 - b$ or $z = 1 - d$. It follows that $1 - d = 1 - b$ which means that $b = d$.

Similarly we show that $a = b$, $c = d$, which implies $a = b = c = d$, and $x = y = w = z$.

Therefore $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the only pair that is not compatible.

Solution 6 (David Hunt)

The AM-GM inequality is

$$\sqrt{ab} \leq \left(\frac{a+b}{2}\right)^2 \quad \text{or} \quad ab \leq \frac{a+b}{2}.$$

If n is even

$$\begin{aligned} n! &= (1.n)(2.n-1)\ldots\left(\frac{n}{2}.\frac{n}{2}+1\right) \\ &\leq \left(\frac{n+1}{2}\right)^2\left(\frac{n+1}{2}\right)^2\cdots\left(\frac{n+1}{2}\right)^2 = \left(\frac{n+1}{2}\right)^n. \end{aligned}$$

For n odd the proof is similar. Next,

$$\begin{aligned} \frac{(2n)!}{n!} &= (n+1.2n)(n+2.2n-1)\ldots \\ &\leq \left(\frac{3n+1}{2}\right)^n, \end{aligned}$$

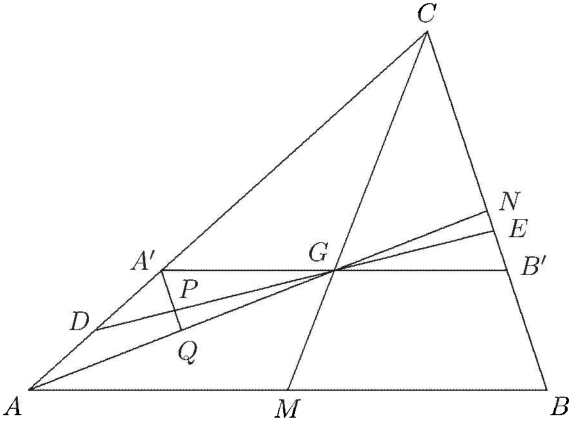
$$\frac{(3n)!}{n!} \le \left(\frac{5n+1}{2}\right)^n, \quad \text{etc.}$$

Hence

$$\begin{aligned}(1998)! &= \frac{(1998)!}{(1997)!}\cdot\frac{(1997)!}{(1996)!}\cdots\frac{(2n)!}{n!}n! \\ &\le \left(\frac{3995n+1}{2}\right)^n\left(\frac{3993n+1}{2}\right)^n\cdots\left(\frac{3n+1}{2}\right)^n\left(\frac{n+1}{2}\right)^n \\ &= \left(\frac{3995n+1}{2}\cdot\frac{3993n+1}{2}\cdots\frac{3n+1}{2}\cdot\frac{n+1}{2}\right)^n\end{aligned}$$

as required.

Solution 7



Let A' and B' be points on the sides AC and BC respectively such that $A'B'$ is parallel to AB and G lies on $A'B'$. Let M be the midpoint of AB.

Since G is the centroid of triangle ABC, $CG : CM = 2 : 3$. Hence $|A'B'C| = \frac{4}{9}|ABC|$ since the triangles $A'B'C$ and ABC are similar. Hence the area of the quadrilateral $AA'B'B$ equals $\frac{5}{9}|ABC|$. Therefore

$$|AA'B'B| - |A'B'C| = \frac{1}{9}|ABC| = 222 \text{ cm}^2.$$

Next we shall show that $A_1 - A_2$ cannot exceed 222 cm^2.

Draw a line through G that intersects AC and BC in the points D and E, say, respectively. Without loss of generality, we can assume that D is strictly between A and A'. Then E is between C and B'. If N is the midpoint of BC, G lies on AN as G is the centroid of triangle ABC. Let Q be the point on AN such that $A'Q$ is parallel to CB. Also let $A'Q$ intersect DE at P.

Since $A'G = B'G$, the triangles $A'PG$ and $B'EG$ are congruent and also the triangles PGQ and EGN are congruent. It follows that $|DA'G| > |EB'G|$ and $|ADG| > |NEG|$. Hence

$$\frac{1}{2}|ABC| = |ACN| > |DCE| > |A'CB'| = \frac{4}{9}|ABC|,$$

which implies that $A_1 - A_2 < \frac{1}{9}|ABC|$ for the line DE.

Thus the answer is 222 cm^2.

Solution 8

Let X be one of the Udakakan cities. Let $T_1, T_2, \ldots, T_m$ be the trading partners of X and $N_1, N_2, \ldots, N_k$ the cities with which X had no trade relations. Now every pair T_i, T_j has another trading partner beside X by condition (a) and that trading partner must be some N_l by condition (b). On the other hand, every N_i has trade relations with exactly two of the T_j's by condition (a). It follows that $k = \binom{m}{2}$ and the total number of Udakakan cities is $1 + m + \binom{m}{2}$. This number lies between 12 and 28 only if $m = 5$ or $m = 6$ and is therefore either 16 or 22.

Let us label the common trading partner of T_i and T_j, $i < j$, other than X, as N_{ij}.

Consider first the case $m = 6$. Since any city can be chosen to play the role of X, every city had exactly 6 trading partners. Two of the trading partners of N_{12} are T_1 and T_2. The other four trading partners must be among N_{34}, N_{35}, N_{36}, N_{45}, N_{46} and N_{56}. Without loss of generality, we may assume that N_{12} and N_{34} have no trade relations. But then by condition (a), they ought to have two trading partners in common while only N_{56} qualifies. This contradiction shows that $m = 6$ is impossible.

Now consider the case $m = 5$. It can be checked that if exactly the following pairs of cities had a trade relationship:
(X, T_i), $i = 1, 2, \ldots, 5$;
(T_i, N_{ij}), $1 \leq i < j \leq 5$;
(T_j, N_{ij}), $1 \leq i < j \leq 5$;
(N_{ij}, N_{kl}), $1 \leq i < j \leq 5$, $1 \leq k < l \leq 5$ and $\{i, j\} \cap \{k, l\} = \emptyset$,
then all the conditions are satisfied.

The archaeologists thus concluded that there were exactly 16 Udakakan cities.

RESULTS

Number of Candidates: 134

High Distinction

Ghan Justin	Pembroke School	SA	12
Cheeseman Andrew	Mentone Grammar School	VIC	11
Farrar Stephen	James Ruse Agricultural High School	NSW	12
Chu Geoffrey	Scotch College	VIC	10
Miyazaki Hiroshi	Scotch College	VIC	12
Irrgang Steven	Caringbah High School	NSW	12
Sun Kevin	James Ruse Agricultural High School	NSW	11
Varodayan David	Sydney Grammar School	NSW	12
Koonin Justin	Sydney Grammar School	NSW	12
Ramchen Lev	Melbourne Grammar School	VIC	10

Distinction

Sewell Thomas	James Ruse Agricultural High School	NSW	10
Ganesan Prasad	Sydney Grammar School	NSW	12
Wong Ka-Shu	James Ruse Agricultural High School	NSW	11
Moase Will	The Hutchins School	TAS	12
Sim Ie-Wen	St Joseph's College	QLD	12
Zhang Geordie	Melbourne Grammar School	VIC	11
Xia Thomas	James Ruse Agricultural High School	NSW	10
Ho Hai-Trung	Sefton High School	NSW	12
Wong Brendan	St Peter's College	SA	11
Ho Gladys	James Ruse Agricultural High School	NSW	12
Lo Jason	Mt Gravatt High School	QLD	11
Harris Martin	Lilydale High School	VIC	12
Ho Frederick	St Ignatius College	NSW	12
Rowe Kieran	Anglican Church Grammar School	QLD	11
Mitra Joydeep	James Ruse Agricultural High School	NSW	11
Sly Allan	Radford College	ACT	10
Tong Kester	Lyneham High School	ACT	10
Chan Kim Kuan	Sydney Boys High School	NSW	12

Pearce Nathan	All Saints Merrimac	QLD	12
Chan Richard	Unley High School	SA	12
Shaw Thanom	Baulkham Hills High School	NSW	11
Cheung Gregory	James Ruse Agricultural High School	NSW	12
Qin Jim	Girraween High School	NSW	12
Solomon Adam	Sydney Boys High School	NSW	12
Anderson Kim	St Peter's College	SA	11
Marshall Robert	University High School	VIC	11

Australian Team to participate in 1998 IMO at Taipei, Taiwan

Justin Ghan, Andrew Cheeseman, Stephen Farrar, Geoffrey Chu, Hiroshi Miyazaki, Justin Koonin.

1999

Problems

PAPER 1 Tuesday 09 February

1. Points P, Q, R and S lie, in that order, on a circle such that PQ is parallel to SR and $QR = SR$. The point T lies in the same plane as the circle, QT is a tangent to the circle and the angle RQT is acute. Prove that

(a) $PS = QR$;

(b) angle PQT is trisected by QR and QS.

2. A town has 99 clubs $C_1, C_2, \ldots, C_{99}$, each of which has at least one member and no two of which have exactly the same members.

Determine the smallest positive integer n such that one can be certain there is a set S of n people with the property:

whenever C_i and C_j, $1 \leq i, j \leq 99$, are different clubs in the town, then there is either a person in S who belongs to C_i but not to C_j, or there is a person in S who belongs to C_j but not to C_i.

3. (a) Find positive integers a_1, a_2, a_3 and d_3 such that

(i) $a_k - a_{k-1} = d_3$ for $k = 2, 3$, and

(ii) there are integers $m_i > 1$ and b_i such that $a_i = b_i^{m_i}$ for $i = 1, 2, 3$.

(b) Show that for each integer $n > 1$ there exist positive integers $a_1, a_2, \ldots, a_n$ and d_n such that

(i) $a_k - a_{k-1} = d_n$ for $k = 2, 3, \ldots, n$, and

(ii) there are integers $m_i > 1$ and b_i such that $a_i = b_i^{m_i}$ for $i = 1, 2, \ldots, n$.

4. In the triangle $\triangle$ the radius of the incircle is r.

Prove that the sum of the lengths of the altitudes of $\triangle$ is at least $9r$.

PAPER 2 Wednesday 10 February

5. Let $x > 1$ be a real number and $n > 1$ an integer.

Prove that

$$1 + \frac{x-1}{nx} < \sqrt[n]{x} < 1 + \frac{x-1}{n}.$$

6. Let ABC be a triangle and D, E, F points exterior to the triangle such that $\triangle ABD$, $\triangle BCE$ and $\triangle CAF$ are equilateral. The sides of these triangles are extended to produce the following intersections: BE and AF intersect in K, DB and FC intersect in L, and DA and EC intersect in M.

Prove that DK, EL and FM are parallel.

7. Let n be an integer and p a prime number such that $1 + np$ is a perfect square.

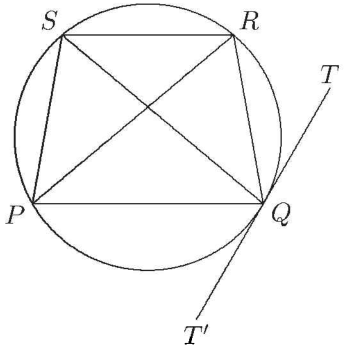
Prove that $n + 1$ is the sum of p perfect squares.

8. (a) Find one sequence $a_1, a_2, a_3, \ldots$ of integers with the following properties:

- (i) $a_n = 1$ or -1 for each n;
- (ii) $a_{mn} = a_m a_n$ for all m and all n;
- (iii) for no value of n does $a_n = a_{n+1} = a_{n+2}$ hold.

(b) Determine all sequences $a_1, a_2, a_3, \ldots$ of integers with properties (i), (ii) and (iii).

Solution 1 (Geoffrey Chu, Scotch College, VIC)



T is on the same side of QS as R. For suppose otherwise and let the point be T' with $\angle T'QR$ acute. Then

$$\begin{aligned} \angle QRP &= \angle T'QP < 90^\circ \quad \text{(alternate segment)} \\ \text{and} \quad \angle PRS &= \angle PQS, \quad \text{(same segment)} \end{aligned}$$

hence

$$\begin{aligned} \angle QRS &= \angle QRP + \angle PRS \\ &= \angle T'QP + \angle PQS \\ &= \angle T'QS < 90^\circ, \\ \angle PQR &< \angle T'QR < 90^\circ, \\ \text{and} \quad \angle QRS + \angle PQR &< 180^\circ, \end{aligned}$$

contradicting $PQ \parallel RS$.

Next let $\angle QPR = \alpha$. Then $\angle QSR = \alpha$ ($PQRS$ cyclic), $\angle SQR = \alpha$ (isosceles triangle), $\angle TQR = \alpha$ (alternate segment) and $\angle PQS = \alpha$ ($PQ \parallel RS$, alternate angles).

We now have $\alpha = \angle TQR = \angle RQS = \angle SQP$ so angle PQT is trisected by QR and QS.

Finally, $\angle SPR = \angle SQR = \alpha$ (same segment) so $\triangle PSR$ is isosceles and $PS = SR = QR$, as required.

Solution 2 (Andrew Cheeseman, Mentone Grammar School, VIC)

The answer is $|S| = n = 98$. We will show by an example that $n \geq 98$ and then show $n \leq 98$ by strong induction on the number of clubs in the town.

First consider a town with 99 people, each in their own club with no other members. If there were a set S with 97 members then at least 2 clubs C_1, C_2 would have no members in S. Hence S has no member in C_1 which is not in C_2 and vice versa. Thus $n \geq 98$.

Next consider a town with exactly 2 clubs, C_1, C_2 which satisfy the given conditions. That is, $C_1 \neq C_2$. Now since $C_1 \neq C_2$ the clubs have one member x not in both of them. Let $S = \{x\}$. Then the size of S is $1 = 2 - 1$ and S satisfies the condition – either x is in C_1, but not C_2 or vice versa. (Clearly, if the town has at least one club we may take S to be the empty set and $|S| = 0 = 1 - 1$.)

We now assume that if a town has 1 or 2 or ... or k clubs with $C_i \neq C_j$ for all $i \neq j$ then we can find a set S with $0, 1, 2, \ldots, k-1$ members satisfying "the condition" – given $C_i \neq C_j$ there is an x in C_i, but not C_j, or vice versa.

Now consider a town with $k+1$ clubs, $C_1, C_2, \ldots, C_{k+1}$, satisfying $C_i \neq C_j$ for all $i \neq j$. Since $C_1 \neq C_2$ there is a person, call him Fred, who is in C_1 but not C_2 or who is in C_2 but not C_1. Let X be the set of clubs Fred is in, X' the set of clubs Fred is not in. $|X| \geq 1$ since Fred is in at least one club and $|X'| \geq 1$ since Fred is not in one club. But $|X| + |X'| = k+1$ hence $|X| \leq k$, $|X'| \leq k$.

Since the size of $X \leq k$ there is a set S_X with $\leq k-1$ members satisfying "the condition" in the "town" with clubs in X only. Similarly there is a set $S_{X'}$ with $|X'| - 1$ members satisfying "the condition" for clubs in X'. Finally consider $S = S_X \cup S_{X'} \cup \{\text{Fred}\}$.

$$|S| \leq |S_X| + |S_{X'}| + 1 \leq |X| - 1 + |X'| - 1 + 1 = |X| + |X'| - 1 = k + 1 - 1 = k.$$

Consider 2 clubs C_i, C_j. If $C_i \in X$ and $C_{X'} \in X'$ then Fred is in C_i but not C_j. If $C_i, C_j \in X$ then there is $x \in S_X$ such that $x \in C_i$ but not C_j or vice versa. Finally if $C_i, C_j \in X'$ then there is $x \in S_{X'}$, with the required property. The proof by induction is complete. So if $k+1 = 99$, $|S| = n < 98$.

Solution 3

Alternative 1 (Kevin Sun, James Ruse Agricultural High School, NSW; Kester Tong, Lake Ginninderra College, ACT)

(a) There are many possible solutions. $a_1 = 1$, $a_2 = 25$, $a_3 = 49$, $d_3 = 24$ is the most obvious; 49, 196, 343 is another.

(b) Proof by induction on n. For $n = 2$, take $a_1 = 1$, $a_2 = 4$, and $d_2 = 3$. Now let $n > 2$ and assume there are positive integers $c_1, c_2, \ldots, c_{n-1}$ and c such that $c_k - c_{k-1} = e$ for $k = 2, 3, \ldots, n-1$ and $c = f_i^{m_i}$ for $i = 1, 2, \ldots, n-1$ and integers f_i, $m_i > 1$.

Let m be the least common multiple of $m_1, m_2, \ldots, , m_{n-1}$ and let $c = c_{n-1} + e$. Put $a_i = c_i c^m$, $i = 1, 2, \ldots, n-1$ and $a_n = c^{m+1}$.

Then $a_i = f_i^{m_i} c^m = (f_i d)^{m_i}$ where $d^{m_i} = c^m$. Also

$$a_{i+1} - a_i = f_{i+1}^{m_{i+1}} c^m - f_i^{m_i} c^m = ec^m$$

and

$$a_{n+1} - a_n = c^{m+1} - c_{n-1} c^m = (c - c_{n-1})\, c^m = ec^m.$$

So $a_1, a_2, \ldots, a_n$ and $d = c^m e$ as required.

Alternative 2 (Andrew Cheeseman, Mentone Grammar School, VIC)

(a) Let

$$\begin{aligned} a_1 &= 2^{20}.3^{24}, \\ a_2 &= 2a_1 = 2^{21}.3^{24} = \left(2^7.3^8\right)^3, \\ a_3 &= 3a_1 = 2^{20}.3^{25} = \left(2^4.3^5\right)^5. \end{aligned}$$

Clearly since $a_2 = 2a_1$, $a_3 = 3a_1$, $d_3 = a_1$ and also condition (ii) is satisfied.
This solution is the first case of the general construction in (b).

(b) If $n \geq 3$ we will show there is an x such that

$$a_k = kx = (b_k)^{f(k)}, \qquad 1 \leq k \leq n,$$

where $f(k)$ is the kth prime, $f(1) = 2$, $f(2) = 3$, $f(3) = 5$, etc. That is, we will construct x such that

$$x = b_1^2, \quad 2x = b_2^3, \quad 3x = b_3^5, \quad 4x = b_4^7, \quad \ldots.$$

Suppose

$$x = 2^{\alpha_1} 3^{\alpha_2} 5^{\alpha_3} 7^{\alpha_4} \ldots f(n)^{\alpha_n},$$

that is, x is a product of (some of) the first n primes. Let us call the equation $a_k = kx = b_k^{f(k)}$ equation (k).

Now equation (1) tells us x is a square and hence α_i, $1 \leq i \leq n$, is even or $\alpha_i \equiv 0 \pmod 3$, $2 \leq i \leq n$. Equation (3) tells us $3x$ is a fifth power so $\alpha_1 \equiv 0 \pmod 5$, $\alpha_2 \equiv 4 \pmod 5$ and $\alpha_i \equiv 0 \pmod 5$, $3 \leq i \leq n$.

Now if $1 \leq k \leq n$, each prime dividing k is less than $f(n)$, the nth prime, so no extra primes intervene. Hence, for all $1 \leq i \leq n$, we obtain congruences for α_i, mod 2, mod 3, mod 5, …, mod $f(n)$. So α_i is determined modulo $2.3.5.7 \ldots f(n)$, by the Chinese

Remainder Theorem. The smallest x is obtained by taking the smallest positive values of α_1, α_2,...,α_n.

For example, if $n = 4$, then $\alpha_1 \equiv 0 \pmod 2$, $\alpha_1 \equiv 2 \pmod 3$, $\alpha_1 \equiv 0 \pmod 5$, $\alpha_1 \equiv 5 \pmod 7$, $\alpha_1 \equiv 110 \pmod{210}$; $\alpha_2 \equiv 0 \pmod 2$, $\alpha_2 \equiv 0 \pmod 3$, $\alpha_2 \equiv 4 \pmod 5$, $\alpha_2 \equiv 0 \pmod 7$, $\alpha_2 \equiv 84 \pmod{210}$; $\alpha_3 \equiv 0 \pmod{210}$; $\alpha_4 \equiv 0 \pmod{210}$ so we may take

$$x = 2^{110}.3^{84}.5^{210}.7^{210}.$$

Alternative 3 (Thomas Sewell, James Ruse Agricultural High School, NSW)

(b) Let $p_1, p_2, \ldots, p_n$ be n distinct primes. Define t_i, $1 \le i \le n$ by

$$\begin{array}{rcrl} t_i & \equiv & -1 & \pmod{p_i} \\ t_i & \equiv & 0 & \pmod{p_j} \quad \text{if } j \neq i. \end{array}$$

By Chinese Remainder Theorem t_i exists, modulo $p_1 p_2 \ldots p_n$. Let

$$d_n = 1^{t_1} 2^{t_2} 3^{t_3} \ldots n^{t_n} = \prod_{i=1}^{n} i^{t_i},$$

and let

$$a_j = j d_n, \qquad 1 \le j \le n.$$

Now clearly $a_{j+1} - a_j = d_n$ and

$$a_j = j d_n = 1^{t_1} 2^{t_2} \ldots j^{t_j} \ldots n^{t_n}.$$

which is a p_jth power by construction.

Solution 4

Suppose the triangle has vertices A, B, C; side lengths a, b, c; area S; lengths of altitudes h_a, h_b, h_c; perimeter $p = a + b + c$ and inradius r. Now

$$2S = rp = ah_a = bh_b = ch_c$$

using area = semi-perimeter × inradius and half base × height. Hence

$$\begin{aligned} h_a &= \frac{rp}{a}, \quad h_b = \frac{rp}{b}, \quad h_c = \frac{rp}{c} \\ h_a + h_b + h_c &= r(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\ &\geq 9r \quad \text{by Cauchy-Schwarz or AM-HM} \end{aligned}$$

or

$$\begin{aligned} &= r\left(3+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{a}{c}+\frac{c}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)\right) \\ &\geq 9r \quad \text{using } x+\frac{1}{x}\geq 2. \end{aligned}$$

Clearly equality occurs iff the triangle is equilateral.

Solution 5

Alternative 1 (found by a number of students)

If $x > 1$

$$1+\frac{x-1}{n}=\frac{x+n-1}{n}=\frac{x+1+\cdots+1}{n}>\sqrt[n]{x} \quad \text{by AM-GM}$$

(there are $n-1$ 1s in the sum).

Also

$$\sqrt[n]{x}>\frac{n}{\frac{1}{1}+\frac{1}{1}+\cdots+\frac{1}{1}+\frac{1}{x}}=\frac{n}{n-1+\frac{1}{x}}=\frac{nx}{nx-x+1} \quad \text{by GM-HM}$$

We will show

$$\frac{nx}{nx-x+1}>1+\frac{x-1}{n}.$$

For consider

$$\begin{aligned} n^2x^2 &> n^2x^2-(x-1)^2 \quad \text{since } x\neq 1, \\ \text{i.e.} \quad n^2x^2 &> (nx+x-1)(nx-x+1), \\ \text{i.e.} \quad \frac{nx}{nx-x+1} &> \frac{nx+x-1}{nx}=1+\frac{x-1}{nx}. \end{aligned}$$

So we have

$$1+\frac{x-1}{n}>\sqrt[n]{x}>\frac{nx}{nx-x+1}>1+\frac{x-1}{nx}$$

as required.

Alternative 2 (provided by Problems Committee)

Consider

$$a^n-1=(a-1)(a^{n-1}+a^{n-2}+\cdots+a+1).$$

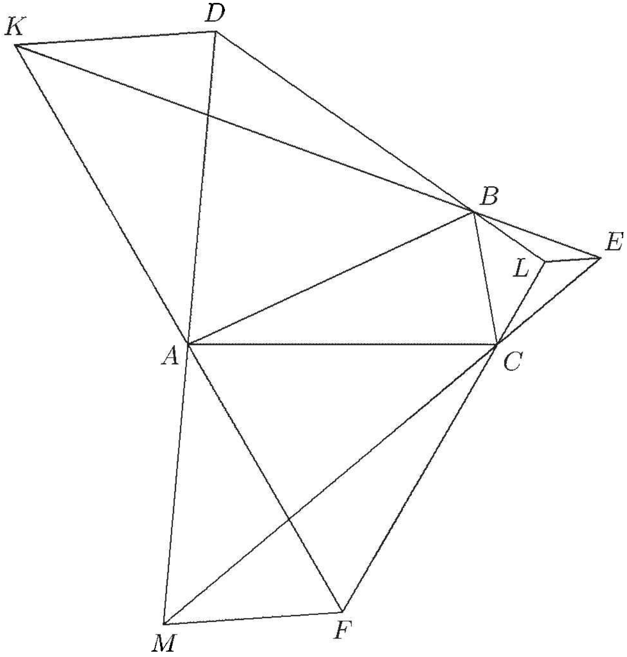
Set $a=\sqrt[n]{x}$ the

$$x-1=\left(\sqrt[n]{x}-1\right)\left(\sqrt[n]{x}^{n-1}+\sqrt[n]{x}^{n-2}+\cdots+\sqrt[n]{x}+1\right).$$

Since $x > 1$, $1 < \sqrt[n]{x^{n-k}} < x$ for $k = 1, 2, \ldots, n-1$. Hence

$$\begin{aligned} n &< \sqrt[n]{x^{n-1}} + \cdots + \sqrt[n]{x} + 1 < nx, \\ \text{i.e.}\quad \frac{1}{nx} &< \frac{1}{\sqrt[n]{x^{n-1}} + \cdots + \sqrt[n]{x} + 1} < \frac{1}{n}, \\ \text{i.e.}\quad \frac{1}{nx} &< \frac{\sqrt[n]{x} - 1}{x - 1} < \frac{1}{n}, \\ \text{i.e.}\quad \frac{x-1}{nx} &< \sqrt[n]{x} - 1 < \frac{x-1}{n}, \\ \text{i.e.}\quad 1 + \frac{x-1}{nx} &< \sqrt[n]{x} < 1 + \frac{x-1}{n}. \end{aligned}$$

Solution 6 (Geoffrey Chu, Scotch College, VIC; Kevin Sun, James Ruse Agricultural High School, NSW; Peter McNamara, Hale School, WA; and several other students)



We will prove that $FM \parallel EL$, then by symmetry of argument, $EL \parallel DK$ as required. Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Then

$$\begin{aligned} \angle LBC &= 180^\circ - \angle DBA - \angle ABC = 120^\circ - \beta \\ \angle LCB &= 180^\circ - \angle BCA - \angle ACF = 120^\circ - \gamma. \end{aligned}$$

Hence

$$\begin{aligned}\angle BLC &= 180^\circ - \angle LBC - \angle LCB\\ &= \beta + \gamma - 60^\circ\\ &= 180^\circ - \alpha - 60^\circ\\ &= 120^\circ - \alpha.\end{aligned}$$

By symmetry of argument

$$\angle ACM = 120^\circ - \gamma, \quad \angle CMA = 120^\circ - \beta, \quad \angle MAC = 120^\circ - \alpha.$$

Hence

$$\triangle CLB \sim \triangle MCF \quad \text{(AAA)}$$

and

$$\begin{aligned}\frac{CB}{CL} &= \frac{CM}{CA}\\ \text{and} \quad \frac{CE}{CL} &= \frac{CM}{CF} \quad \text{(equilateral triangles).}\end{aligned}$$

Also

$$\angle ECL = \angle MCF$$

so

$$\triangle ECL \sim \triangle MCF \quad \text{SAS}$$

and $EL \parallel FM$ as required.

Note: The wording of the question was unfortunate, since for the result to be true D and C must be on the opposite side of AB, etc.

Solution 7

Now by assumption, $1 + np = m^2$ for some positive integer m. Hence $np = m^2 - 1 = (m-1)(m+1)$. Since p is a prime number, it divides $m - 1$ or $m + 1$.

Case 1. Suppose p divides $m - 1$. Then $m - 1 = tp$ and $np = tp(tp + 2)$ or $n = pt^2 + 2t$, $n + 1 = (p-1)t^2 + (t+1)^2$ and $n+1$ is a sum of p perfect squares.

Case 2. Suppose p divides $m + 1$. Then $m + 1 = tp$, $np = tp(tp - 2)$, $n = t^2p - 2t$, $n + 1 = (p-1)t^2 + (t-1)^2$, as required.

Solution 8 (Geoffrey Chu, Scotch College, VIC)

We claim that there are only two such sequences. We define $\{b_n\}$ and $\{c_n\}$ as follows:

$$\begin{array}{rrcrl} & b_n & = & 1 & \text{if } n \equiv 1 \pmod 3 \\ & b_n & = & -1 & \text{if } n \equiv 2 \pmod 3 \\ & b_3 & = & 1 & \\ \text{and} & b_n & = & b_q & \text{if } n = 3^k q \text{ and } (3,q) = 1. \end{array}$$

$$\begin{array}{rrcrl} & c_n & = & 1 & \text{if } n \equiv 1 \pmod 3 \\ & c_n & = & -1 & \text{if } n \equiv 2 \pmod 3 \\ & c_3 & = & -1 & \\ \text{and} & c_n & = & (-1)^k c_q & \text{if } n = 3^k q \text{ and } (3,q) = 1. \end{array}$$

Clearly the condition (i) is satisfied for both and since the sequences are $1, -1, \pm 1, 1, -1, \pm 1, \ldots$ the condition (iii) is satisfied. To check condition (ii) is a little more tricky. Suppose $(mn, 3) = 1$ then $b_{mn} = b_m b_n$ since $1.1 \equiv 2.2 \equiv 1 \pmod 3$ and $1.2 \equiv 2.1 \equiv 2 \pmod 3$. If $m = 3^\alpha p$, $n = 3^\beta p$ then

$$\begin{array}{rrcl} & b_{mn} & = & b_{pq} = b_p b_q = b_m b_n \\ \text{and} & c_{mn} & = & (-1)^{\alpha+\beta} c_{pq} = (-1)^\alpha c_p (-1)^\beta c_q = c_m c_n \end{array}$$

and condition (ii) is satisfied.

To prove that these are the only two sequences satisfying the conditions it is sufficient to prove that any such sequence $\{a_n\}$ that satisfies all three conditions must also have

$$a_n = +1 \quad \text{if } n \equiv 1 \pmod 3 \quad \text{and} \quad a_n = -1 \quad \text{if } n \equiv 2 \pmod 3. \qquad (1)$$

Since then, after you define a_3 as $+1$ or -1 every other term is determined by (ii) and either $a_n = b_n$ for all n or $a_n = c_n$ for all n. We will use strong induction to prove (1).

First $a_1 = (a_1)^2 = 1$. Suppose $a_2 \neq -1$, that is, $a_2 = 1$. Then $a_3 = -1$ or else $a_1 = a_2 = a_3 = 1$.

$$\begin{array}{rrcl} & a_4 & = & a_2 a_2 = 1 \\ & a_6 & = & a_2 a_3 = -1 \\ & a_8 & = & a_2 a_4 = 1 \\ & a_9 & = & a_3 a_3 = 1 \\ & a_7 & = & -1 \quad \text{or else } a_7 = a_8 = a_9 \\ & a_5 & = & 1 \quad \text{or else } a_5 = a_6 = a_7 \\ & a_{10} & = & a_2 a_5 = 1 \\ \text{and now} & a_8 & = & a_9 = a_{10} = 1. \end{array}$$

Hence we have shown $a_1 = 1$, $a_2 = -1$.

Next we assume $a_{3k+1} = 1$, $a_{3k+2} = -1$, $1 \leq k \leq n-1$. There are 4 cases, $a_{3n+1} = \pm 1$, $a_{3n+2} = \pm 1$.

Case 1. $a_{3n+1} = a_{3n+2} = 1$.

Then

$$\begin{aligned} a_{3n} &= a_{3n+3} = -1 \quad \text{by (iii)} \\ a_{6n} &= a_{3n}a_2 = 1 \\ a_{6n+6} &= a_{3n+3}a_2 = 1 \\ a_{6n+2} &= a_{3n+1}a_2 = -1 \\ a_{6n+4} &= a_{2n+2}a_2 = -1. \end{aligned}$$

Thus

$$a_{6n+3} = 1 \quad \text{or else} \quad a_{6n+2} = a_{6n+3} = a_{6n+4}.$$

Hence

$$a_{6n} = a_{6n+3} = a_{6n+6} = 1$$

and

$$a_{2n} = a_{2n+1} = a_{2n+2} = \frac{1}{a_3}$$

and (iii) is contradicted.

Case 2. $a_{3n+1} = a_{3n+2} = -1$ can be dealt with in the same way as *Case 1.*

Case 3. $a_{3n+1} = -1$, $a_{3n+2} = 1$.

Either $3n+1$ or $3n+2$ is even, hence composite. Suppose $3n+1 = bc$ where either $b \equiv c \equiv 1 \pmod 3$ or $b \equiv c \equiv 2 \pmod 3$. Then, by induction, $a_b = a_c$ and $a_{3n+1} = a_b a_c = +1$, a contradiction. Similarly, if $3n+2 = bc$, we have $a_b \neq a_c$ and $a_{3n+2} = -1$.

Hence the 4th case $a_{3n+1} = 1$ and $a_{3n+2} = -1$ must occur and the induction proof is complete. So (1) is true for all n and the two given sequences are the only ones satisfying (i), (ii) and (iii).

RESULTS

Number of Candidates: 135

High Distinction

Chu Geoffrey	Scotch College	VIC	11
Sun Kevin	James Ruse Agricultural High School	NSW	12
Cheeseman Andrew	Mentone Grammar School	VIC	12
Lo Jason	Mt Gravatt High School	QLD	12
Xia Thomas	James Ruse Agricultural High School	NSW	11
Tong Kester	Lake Ginninderra College	ACT	11
McNamara Peter	Hale School	WA	10
Ramchen Lev	Melbourne Grammar School	VIC	11
Rowe Keiran	Anglican Church Grammar School	QLD	12
Zhang Geordie	Melbourne Grammar School	VIC	12
Sewell Thomas	James Ruse Agricultural High School	NSW	11

Distinction

Lu Yi-Hua	Lyneham High School	ACT	10
Mitra Joydeep	James Ruse Agricultural High School	NSW	12
Shaw Thanom	Baulkham Hills High School	NSW	12
Easton Brett	Lake Ginninderra College	ACT	12
Thai Duc	Rostrevor College	SA	10
Vukcevic Damjan	Caulfield Grammar School	VIC	11
Sly Allan	Radford College	ACT	11
James Clancy	Westminster School	SA	11
Kwok Andrew	University High School	VIC	9
Wong Ka-Shu	James Ruse Agricultural High School	NSW	12
Delprado Anton	North Sydney Boys High School	NSW	12
Yan Mu	Mount Gravatt High School	QLD	12
Mathews Peter	Prince Alfred College	SA	10
Tan Lit Hau	James Ruse Agricultural High School	NSW	10
Rao Gautam	Mentone Grammar School	VIC	9
Schuster Grant	Pulteney Grammar School	SA	12

Zhao Yiying	Penleigh and Essendon Grammar School	VIC	9
New Elizabeth	MLC School, Burwood	NSW	11
Tsai Tony	James Ruse Agricultural High School	NSW	11
Edwards Catherine	St Hilda's Anglican School for Girls	WA	12
Shin Edward	Melbourne Grammar School	VIC	12
Lam Deborah	Hornsby Girls High School	NSW	12
Tsui Kenneth	Bossley Park High School	NSW	11
Griffin Damien	John Paul College Coffs Harbour	NSW	11
Pearce Garth	Penrith High School	NSW	11
Chan Vincent	Sydney Boys High School	NSW	11
Lee Jessica	James Ruse Agricultural High School	NSW	12
Kameron Matthew	Knox Grammar School	NSW	12

Australian Team to participate in 1999 IMO at Bucharest, Romania

Geoffrey Chu, Kevin Sun, Andrew Cheeseman, Kester Tong, Peter McNamara, Allan Sly.

2000

Problems

PAPER 1 Tuesday 08 February

1. Find all polynomials f with real coefficients such that

$$(x-27)f(3x) = 27(x-1)f(x)$$

for every real number x.

2. For each date of the current year (2000) we evaluate the expression

$$\text{day}^{\text{month}} - \text{year}$$

and then find the highest power of 5 dividing it.

(For example, the 293rd anniversary of Leonhard Euler's birth will be on 15 April, in which case we obtain:

$$\text{day}^{\text{month}} - \text{year} = 15^4 - 2000 = 5^3(405-16) = 5^3 \cdot 389,$$

a multiple 5^3 but not of 5^4.)

Find all dates for which the corresponding power of 5 is the greatest.

3. Let $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ be real numbers such that

(i) $0 < x_1y_1 < x_2y_2 < \ldots < x_ny_n$ and

(ii) $x_1 + x_2 + \cdots + x_i \geq y_1 + y_2 + \cdots + y_i$ for $1 \leq i \leq n$.

Prove that $\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \leq \frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}$.

When does equality occur?

4. Let A, B, C, A', B', C' be points on a circle such that AA' is perpendicular to BC, BB' is perpendicular to CA, and CC' is perpendicular to AB. Further, let D be a point on that circle and let DA' intersect BC in A'', DB' intersect CA in B'', and DC' intersect AB in C'', all line segments being extended where required.

Prove that A'', B'', C'' and the orthocentre of triangle ABC are collinear.

PAPER 2 Wednesday 09 February

5. Let m and n be positive integers. Prove that $m^{n+2} + m^{n+1} + m^n$ is not a perfect square.

6. Let a, b and c be any real numbers not all of which are zero. Determine all functions f that assign real numbers to real numbers such that

$$af(xy + z^2) + bf(yz + x^2) + cf(zx + y^2) = 0$$

for all real numbers x, y, z.

Distinguish all possibilities for a, b and c.

7. Solve the following system of equations:

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 200.0 \\ \{x\} + y + \lfloor z \rfloor &= 190.1 \\ \lfloor x \rfloor + \{y\} + z &= 178.8. \end{aligned}$$

Note that if r is a real number, then $\lfloor r \rfloor$ denotes the largest integer not exceeding r, while $\{r\}$ stands for the fractional part of r, that is $r - \lfloor r \rfloor$.

8. Let A_1, A_2, A_3, A_4, A_5, A_6, A_7 be vertices of a heptagon lying in the plane H, and let B and C be two different points in space, not lying in H, such that no three of these nine points are collinear. Each of the 14 edges A_iB and A_iC ($i = 1, 2, \ldots, 7$), the 14 diagonals of heptagon $A_1A_2A_3A_4A_5A_6A_7$, and the line segment BC are coloured either green or gold.

Prove that there are three line segments among them, all of the same colour, that form a triangle.

Solution 1

Alternative 1 (Peter Mathews, Prince Alfred College, SA)

Consider the equation

$$(x-27)f(3x)=27(x-1)f(x). \qquad (1)$$

If we set $x=27$ then $0=0\cdot f(81)=27\cdot 26\cdot f(27)$, that is, $f(27)=0$. Similarly, if we set $x=1$ then $0=27\cdot 0\cdot f(1)=-26\cdot f(3)$ so $f(3)=0$ as well.

Hence $f(x)=(x-3)(x-27)q(x)$ which we substitute back into equation (1)

$$(x-27)(3x-3)(3x-27)q(3x)=27(x-1)(x-3)(x-27)q(x)$$

so for $x\neq 1,27$

$$(x-9)q(3x)=3(x-3)q(x). \qquad (2)$$

Next if we set $x=3$, $0=3\cdot 0\cdot q(3)=-6q(9)$ or $q(9)=0$ so we let $q(x)=(x-9)g(x)$ and substituting into equation (2) we obtain

$$(x-9)(3x-9)g(3x)=3(x-3)(x-9)g(x),$$

i.e.

$$g(3x)=g(x), \quad \text{for} \quad x\neq 1,\ 3,\ 9,\ 27.$$

In particular, if we set $x=2$ then $g(2)=g(6)=g(18)=\ldots=g(2\cdot 3^k)$ for all k. Hence either $g(x)=g(2)$ has an infinite number of roots, which is impossible, or $g(x)$ is a constant $=a$, say.

Hence $q(x)=a(x-9)$ and $f(x)=a(x-3)(x-9)(x-27)$. Conversely it is easy to check that all such cubic polynomials satisfy equation (1).

Alternative 2 (Geoffrey Chu, Scotch College, VIC)

Consider

$$(x-27)f(3x)\equiv 27\,(x-1)f(x). \qquad (1)$$

Suppose $f(x)=a_nx^n+\ldots+a_0$ then

$$(x-27)\left(a_n(3x)^n+\ldots+a_0\right)\equiv 27(x-1)(a_nx^n+\ldots+a_0)$$

and equating the coefficients of the leading terms we have $a_n3^n=27\,a_n$ or $n=3$, so f is a cubic polynomial.

We now set $x=27$ in (1) yielding $0=27\cdot 26\cdot f(27)$ so 27 is a root of f. Next we set $x=9$ in (1) and $-18f(27)=27\cdot 8\cdot f(9)$ and hence $f(9)=0$ and 9 is a root of f. Finally we set $x=3$ yielding $0=-24f(9)=$

$27 \cdot 2 \cdot f(3)$ so 3 is a root of f. So we have shown that $(x-3)(x-9)(x-27)$ divides $f(x)$ and since f is a cubic

$$f(x) = A(x-3)(x-9)(x-27), \quad A \in \mathbb{R}.$$

It is easy to check that all such functions satisfy the equation (1).

Solution 2

Alternative 1 (Damjan Vukcevic, Caulfield Grammar School, VIC)

Let $A = \text{day}^{\text{month}}$ and $B = \text{day}^{\text{month}} - \text{year}$ and suppose $A = 5^n \cdot k$ where $(5, k) = 1$. We consider, in turn, $n = 0$, 1, 2, 3 and $n \geq 4$.

If $n = 0$, then $A = k$ and $B = k - 2000 \not\equiv 0 \pmod 5$.

If $n = 1$, then $A = 5k$ and $B = 5(k-400)$ and 5 divides B but 5^2 does not divide B.

If $n = 2$, then $A = 25k$ and $B = 25(k-80)$ so 25 divides B but 125 does not divide B.

If $n = 3$, then $A = 125k$ and $B = 125(k-16)$ so 125 divides B but a larger power of 5 may divide B.

If $n \geq 4$ then $A = 5^n k$ and $B = 5^3(5^{n-3}k - 16)$ so 5^3 exactly divides B.

Clearly we must consider the $n = 3$ case more closely since this may lead to a greater power of 5 than 5^3 dividing B. If $n = 3$ then the month cannot be April or later in the year. It also cannot be February since the power of 5 dividing A would then be even. If the month were January we would need a date divisible by 5^3, which is greater than 31. Hence we only need to consider March and days divisible by 5 but not by 5^2.

5 March	$B = 5^3 - 2000 = 5^3(1-16) = 5^4(-3)$.
10 March	$B = 2^3 5^3 - 2000 = 5^3(8-16) = 5^3(-8)$.
15 March	$B = 3^3 5^3 - 2000 = 5^3(27-16) = 5^3(11)$.
20 March	$B = 4^3 5^3 - 2000 = 5^3(64-16) = 5^3(48)$.
30 March	$B = 6^3 5^3 - 2000 = 5^3(216-16) = 5^3(200) = 5^5(8)$.

Hence there is a unique date with maximal power, 5^5, dividing B and that is March 30th 2000.

Alternative 2 (Geoffrey Chu, Scotch College, VIC)

For 30th March 2000, 5^5 exactly divides $30^3 - 2000 = 25000$. So consider all days d and months m for which the power of 5 dividing $d^m - 2000$ is at least 5^4. That is,

$$5^4 | d^m - 2000 \quad \text{or} \quad d^m - 2000 = t \cdot 5^4, \quad d^m = (5t+16) \cdot 5^3.$$

Hence $5^3 \parallel d^m$. Since 5 is prime $5|d$. If $5^2|d$ then 5^3 is not the exact power of 5 dividing d^m since 3 is odd. So $5 \parallel d$. Thus, $m = 3$ and $d = 5$, 10, 15, 20 or 30. Let $d = 5s$ so $d^3 - 2000 = 5^3(s^3 - 16)$ so we need $s^3 \equiv 1 \pmod 5$ or $s \equiv 1 \pmod 5$. If $s = 1$ then $d^3 - 2000 = 5^4(-3)$ and if $s = 6$ then $d^3 - 2000 = 5^3(216 - 16) = 5^5(8)$.

Hence 30th March is the only date for which 5^5 divides $d^m - 2000$.

Solution 3

Alternative 1 (Thomas Xia, James Ruse Agricultural High School, NSW)

Lemma. (Abel's identity) If $S_k = \sum_{i=1}^{k} a_i$ then

$$\sum_{k=1}^{n} a_k b_k = S_n b_n + \sum_{k=1}^{n-1} S_k(b_k - b_{k+1}).$$

Proof.

$$\begin{aligned} & S_n b_n + \textstyle\sum_{k=1}^{n-1} S_k(b_k - b_{k+1}) \\ = \quad & (S_n - S_{n-1})b_n + \textstyle\sum_{k=1}^{n-1}(S_k - S_{k-1})b_k \\ = \quad & a_n b_n + \textstyle\sum_{k=1}^{n-1} a_k b_k, \qquad \text{as required.} \end{aligned}$$

We assume

$$0 < x_1y_1 < x_2y_2 < \ldots < x_ny_n \tag{1}$$

$$\text{and} \quad x_1 + x_2 + \ldots + x_i \geq y_1 + y_2 + \ldots + y_i \quad \text{for} \quad 1 \leq i \leq n. \tag{2}$$

We want to prove

$$\sum_{i=1}^{n} \frac{1}{x_i} \leq \sum_{i=1}^{n} \frac{1}{y_i} \qquad \text{or} \qquad \sum_{i=1}^{n} \left(\frac{1}{y_i} - \frac{1}{x_i}\right) \geq 0$$

$$\text{or} \qquad \sum_{i=1}^{n} \left(\frac{x_i - y_i}{x_iy_i}\right) \geq 0. \tag{3}$$

We set $a_i = x_i - y_i$ and $b_i = \dfrac{1}{x_iy_i}$, $1 \leq i \leq n$ then by the Lemma

$$\begin{aligned} \sum_{i=1}^{n} \left(\frac{x_i - y_i}{x_iy_i}\right) &= \sum_{i=1}^{n}(x_i - y_i) \frac{1}{x_ny_n} \\ &\quad + \sum_{k=1}^{n-1} \left(\sum_{i=1}^{k}(x_i - y_i)\right) \left(\frac{1}{x_ky_k} - \frac{1}{x_{k+1}y_{k+1}}\right). \end{aligned}$$

By (2) $\sum_{i=1}^{n}(x_i - y_i) \geq 0$ and by (1) $x_n y_n > 0$ or $1/(x_n y_n) > 0$ thus the first term is positive. Similarly, by (2),

$$S_k = \sum_{i=1}^{k}(x_i - y_i) \geq 0$$

and since $x_k y_k < x_{k+1} y_{k+1}$ by (1), that is,

$$\frac{1}{x_k y_k} > \frac{1}{x_{k+1} y_{k+1}}$$

or

$$\frac{1}{x_k y_k} - \frac{1}{x_{k+1} y_{k+1}} > 0.$$

Hence (3) is proved with equality if and only if $\sum_{i=1}^{k}(x_i - y_i) = 0$ for $1 \leq k \leq n$ or $x_i - y_i = 0$ for $1 \leq i \leq n$, that is, $x_i = y_i$, $1 \leq i \leq n$.

Alternative 2 (Geoffrey Chu, Scotch College, VIC; Peter McNamara, Hale School, WA)

We assume

$$0 < x_1 y_1 < x_2 y_2 < \ldots < x_n y_n \tag{1}$$

$$\text{and} \quad x_1 + \ldots + x_i \geq y_1 + \ldots + y_i \quad \text{for} \quad 1 \leq i \leq n. \tag{2}$$

Lemma. For $1 \leq i \leq n$

$$\sum_{j=1}^{i} \frac{1}{y_j} - \sum_{j=1}^{i} \frac{1}{x_j} \geq \frac{\sum_{j=1}^{i} x_j - \sum_{j=1}^{i} y_j}{x_i y_i}.$$

Proof. If $i = 1$ then $\dfrac{1}{y_1} - \dfrac{1}{x_1} = \dfrac{x_1 - y_1}{x_1 y_1}$, as required. So, to prove by induction, assume the result for $i = k$. For $i = k + 1$

$$\sum_{j=1}^{k+1} \frac{1}{y_j} - \sum_{j=1}^{k+1} \frac{1}{x_j} = \frac{1}{y_{k+1}} - \frac{1}{x_{k+1}} + \sum_{j=1}^{k} \frac{1}{y_j} - \sum_{j=1}^{k} \frac{1}{x_j}$$

$$\geq \frac{x_{k+1} - y_{k+1}}{x_{k+1} y_{k+1}} + \frac{\sum_{j=1}^{k} x_j - \sum_{j=1}^{k} y_j}{x_k y_k}$$

by the inductive assumption

$$\geq \frac{x_{k+1} - y_{k+1}}{x_{k+1} y_{k+1}} + \frac{\sum_{j=1}^{k} x_j - \sum_{j=1}^{k} y_j}{x_{k+1} y_{k+1}}$$

(since by (1) $\dfrac{1}{x_k y_k} > \dfrac{1}{x_{k+1}y_{k+1}}$ and by (2) $\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j$.)

$$= \frac{\sum_{j=1}^{k+1}(x_j - y_j)}{x_{k+1}y_{k+1}}, \quad \text{as required.}$$

The case $i = n$ of the lemma gives

$$\sum_{j=1}^{n} \frac{1}{y_j} - \sum_{j=1}^{n} \frac{1}{x_j} \geq \frac{\sum_{j=1}^{n}(x_j - y_j)}{x_n y_n} \geq 0.$$

So

$$\frac{1}{y_1} + \ldots + \frac{1}{y_n} \geq \frac{1}{x_1} + \ldots + \frac{1}{x_n}$$

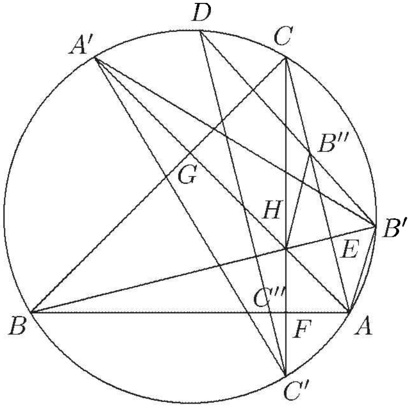
with equality if and only if

$$\sum_{j=1}^{i} x_j - \sum_{j=1}^{i} y_j = 0, \quad 1 \leq i \leq n,$$

since $x_{i+1}y_{i+1} > x_i y_i$, that is, equality if and only if $x_i = y_i$, $\quad 1 \leq i \leq n$.

Solution 4

Alternative 1 (Geoffrey Chu, Scotch College, VIC)



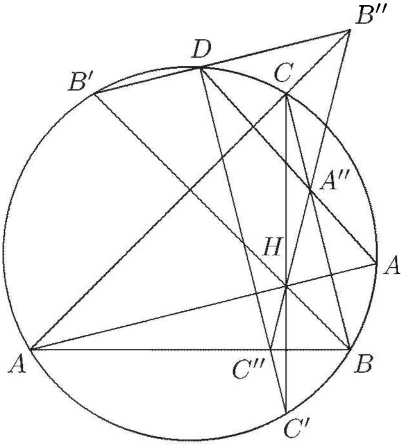
Since AA' is perpendicular to BC it is an altitude of the triangle ABC. The altitudes AA', BB' and CC' meet at H, the orthocentre of the triangle ABC. Let E, F, G be the points where BB' meets AC, CC' meets AB and AA' meets BC.

Let $/GBH = \beta$. Since $/BGH = /AEH = 90°$, the triangles BGH, AEH are similar and $/EAH = \beta$. Since B, C, B', A are concyclic we also have $\angle CAB' = \beta$ (angle in the same segment). Clearly, $\triangle HEA \equiv \triangle B'EA$ (equal angles and common side) so $HE = EB'$. Thus for any point X on AC we have an isosceles triangle $B'HX$; in particular, we have $\angle B''B'H = \angle B''HB'$.

Next let $\angle BAA' = \alpha$, $\angle ABB' = \gamma$ and $\angle A'B'D = \theta$ $(\alpha + \beta + \gamma = 90°)$. Since A', B', A, B are concyclic, $\angle A'B'B = \angle A'AB = \alpha$. So $\angle B''HB' = \angle B''B'H = \alpha + \theta$. Since A', D, B', C' are concyclic we have $\angle A'C'D = \angle A'B'D = \theta$. Since A', C', A, C are concyclic, $\angle A'C'C = \angle A'AC = \beta$ so $\angle C''C'H = \angle A'C'C - \angle A'C'D = \beta - \theta$. Arguing, as in the second paragraph, we have $C'F = FH$ and $C'H$ perpendicular to AB. Thus $C''C'H$ is an isosceles triangle and $\angle C''HF = \beta - \theta$.

Finally $\angle BHF = 90° - \angle HBF = 90° - \gamma$. $\angle BHC'' = \angle BHF - \angle C''HF = 90° - \gamma - (\beta - \theta) = (90° - \beta - \gamma) + \theta = \alpha + \theta$. Thus $\angle B''HE = \angle BHC''$ and C'', H, B'' are collinear (vertically opposite angles). The argument to show C'', H, A'' are collinear is very similar.

Alternative 2 (Problems Committee)



Let H be the orthocentre of the triangle ABC. As AB is perpendicular to CC' and BC is perpendicular to AA', we have $\angle BAA' = \angle BCC'$. Furthermore, angles being subtended by the same chord $C'B$, we have $\angle BCC' = \angle BAC'$. Hence $\angle BAA' = \angle BAC'$.

Join H to A'', B'', C''. Then we have to prove that $\angle BHC'' + \angle BHA'' = 180°$, $\angle BHC'' + \angle BHB'' = 180°$. But $\angle BHC'' = \angle BC'C'' = \angle BC'D$ and $\angle BHA'' = \angle BA'A'' = \angle BA'D$. Since $A'BC'D$ is a cyclic quadrilateral, $180° = \angle BC'D + \angle BA'D = \angle BHC'' + \angle BHA''$.

Also $\angle BHB'' = 180° - \angle B'HB'' = 180° - \angle HB'B'' = 180° - \angle HB'D = 180° - \angle BB'D = 180° - \angle BC'D = 180° - \angle BHC''$, as required.

Alternative 3 (Geoffrey Chu, Scotch College, VIC)

Pascal's Theorem If a hexagon lies on a circle then the 3 points of intersection of opposite sides of the hexagon are collinear. (The proof is well-known, for example use Menelaus' and Ceva's theorems.)

Consider the 6 concyclic points A, A', D, C', C, B then $AA' \cap CC'$, $A'D \cap CB$, $DC' \cap BA$ are collinear. That is, H, A'', C'' are collinear. Similarly, consider B', D, C', C, A, B yielding $B'D \cap AC = B''$, $DC' \cap AB = C''$, $CC' \cap BB' = H$ are collinear.

Solution 5

Consider $m^{n+2} + m^{n+1} + m^n = m^n(m^2 + m + 1)$.

Suppose p is a prime factor of m^n then p divides m, p divides $m^2 + m$ so p does not divide $m^2 + m + 1$. Hence m^n and $m^2 + m + 1$ are coprime, that is $(m^n, m^2 + m + 1) = 1$, so if their product is a perfect square each factor must be.

However $m^2 < m^2 + m + 1 < m^2 + 2m + 1 = (m + 1)^2$ so $m^2 + m + 1$ is not a perfect square and thus $m^{n+2} + m^{n+1} + m^n$ is also not a perfect square.

Solution 6 (Geoffrey Chu, Scotch College, VIC)

Suppose $a, b, c \in \mathbb{R}$ and are not all zero and that

$$a\,f(xy + z^2) + b\,f(yz + x^2) + c\,f(zx + y^2) = 0 \text{ for all real numbers } x, y, z.$$

Case 1: $a + b + c \neq 0$ and, without loss of generality, $a \neq 0$.

If $x = y = z = 0$ then $(a+b+c)f(0) = 0$ or $f(0) = 0$. Next set $x = y = 0$ and $z = t$ then $af(t^2) + bf(0) + cf(0) = 0$ so $f(t^2) = 0$ for all real t, that is, $f(x) = 0$ for all $x \geq 0$. Next set $x = -t$, $y = t$ and $z = 0$ so $af(-t^2) + bf(t^2) + cf(t^2) = 0$ or $af(-t^2) = 0$, that is, $f(-t^2) = 0$, or $f(x) = 0$ for all $x \leq 0$. Together, we have shown $f(x) \equiv 0$.

Case 2: $a + b + c = 0$ and, without loss of generality, $a \neq 0$.

Suppose $t \geq 0$ and let $x = \sqrt{\dfrac{t}{2}} = y = -z$ then $af(t) + (b + c)f(0) = 0$ or $a(f(t) - f(0)) = 0$, that is, $f(t) = f(0) = \alpha \in \mathbb{R}$ for all $t \geq 0$.

Finally, suppose $t \geq 0$ and $x = \sqrt{t}$, $y = -\sqrt{t}$ and $z = 0$ then $af(-t) + bf(t) + cf(t) = 0$, $a(f(-t) - f(t)) = 0$ so $f(-t) = f(t) = \alpha$ for all $t \geq 0$. Taken together, we have shown that $f(t) \equiv \alpha$.

Clearly, if $f(t) = \alpha$, for all t, and $a + b + c = 0$ then we have a solution and if $f(t) = 0$, for all t, then we have a solution for all values of a, b, c.

In summary, if $a + b + c \neq 0$ then $f(t) \equiv 0$ and if $a + b + c = 0$ then $f(t) \equiv \alpha \in \mathbb{R}$.

Solution 7

$$\begin{aligned} x + \lfloor y \rfloor + \{z\} &= 200.0 && (1)\\ \{x\} + y + \lfloor z \rfloor &= 190.1 && (2)\\ \lfloor x \rfloor + \{y\} + z &= 178.8 && (3) \end{aligned}$$

We now form (1) + (2) + (3) yielding

$$2x + 2y + 2z = 568.9 \quad \text{or} \quad x + y + z = 284.45 \qquad (4)$$

(4) − (1) gives $y - \lfloor y \rfloor + z - \{z\} = 84.45$ or $\{y\} + \lfloor z \rfloor = 84.45$. But $0 \le \{y\} < 1$ and $\lfloor z \rfloor \in \mathbb{Z}$ so $\{y\} = 0.45$ and $\lfloor z \rfloor = 84$.

Similarly (4) − (2) gives $\lfloor x \rfloor + \{z\} = 94.35$, so $\lfloor x \rfloor = 94$, $\{z\} = 0.35$; and (4) − (3) gives $\{x\} + \lfloor y \rfloor = 105.65$ so $\{x\} = 0.65$ and $\lfloor y \rfloor = 105$.

Therefore $x = 94.65$, $y = 105.45$ and $z = 84.35$ and it is easy to check that these are a solution to (1), (2) and (3).

Solution 8

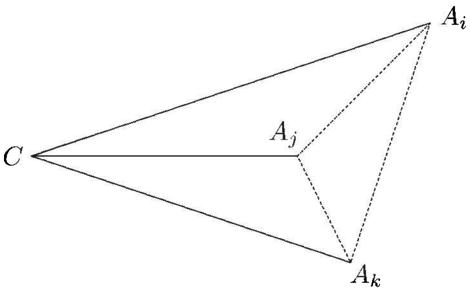
Alternative 1 (Peter McNamara, Hale School, WA)

Without loss of generality BC is coloured green. Assume that it is possible to colour the 14 edges A_iB and A_iC and the 14 diagonals of the heptagon in green and gold so that there does not exist a monochromatic triangle.

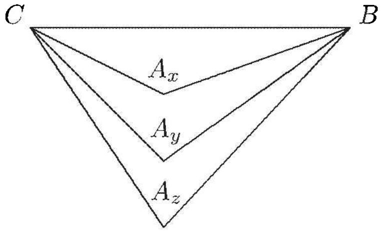
Consider the seven sets

$$\begin{aligned} S_1 &= \{A_1, A_3, A_5\}\\ S_2 &= \{A_2, A_4, A_6\}\\ S_3 &= \{A_3, A_5, A_7\}\\ S_4 &= \{A_4, A_6, A_1\}\\ S_5 &= \{A_5, A_7, A_2\}\\ S_6 &= \{A_6, A_1, A_3\}\\ S_7 &= \{A_7, A_2, A_4\} \end{aligned}$$

Note that the edges of the S_α, $1 \le \alpha \le 7$ are all diagonals of the heptagon. Consider $S_\alpha = \{A_i, A_j, A_k\}$ and suppose that CA_i, CA_j and CA_k are all gold.



Then all of A_iA_j, A_jA_k and A_kA_i are all green, since if not we have a gold triangle, CA_iA_j, say, but now we have a green triangle. Hence if $S_\alpha = \{A_i, A_j, A_k\}$ then one of CA_i, CA_j or CA_k is green, $\alpha = 1, \ldots, 7$. Since each A_i, $1 \le i \le 7$, occurs in exactly three of the sets S_α at least 3 edges CA_x, CA_y and CA_z must be green. Since BC is green, all of BA_x, BA_y and BA_z must be gold.



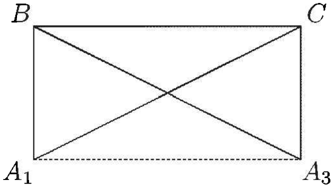
Finally at least one of A_xA_y, A_yA_z and A_zA_x are diagonals of the heptagon, say A_xA_y. If we colour A_xA_y green then CA_xA_y is a green triangle, whereas if we colour it gold then BA_xA_y is a gold triangle, a contradiction.

So no such colouring of the 29 edges is possible.

Alternative 2 (Peter Mathews, Prince Alfred College, SA; Geoffrey Chu, Scotch College, VIC)

Suppose it is possible to colour the edges in such a way that there is no monochromatic triangle. Without loss of generality, let BC be gold. Note that if BA_i is gold then CA_i must be green or else BCA_i is a gold triangle.

Case 1: There are 2 gold lines from B to non-adjacent (that is, non-consecutive) points in the heptagon. So, without loss of generality BA_1 and BA_3 are gold. Hence both CA_1 and CA_3 are green. If A_1A_3 is gold then BA_1A_3 is a gold triangle, if it is green then CA_1A_3 is a green triangle, a contradiction.



Case 2: There are not 2 gold lines from B to non-adjacent points in the heptagon. In this case there are either no gold lines from B or 1 gold line from B or 2 gold lines to adjacent points A_6, A_7, say. Hence BA_1, BA_2, BA_3, BA_4 and BA_5 are all green. Hence each edge of the triangle $A_1A_3A_5$ must be gold (by considering BA_1A_3 etc.). We now have a gold triangle and a final contradiction. Hence there is no such colouring.

RESULTS

Number of Candidates: 125

High Distinction

CHU Geoffrey	Scotch College	VIC	12
XIA Thomas	James Ruse Agricultural High School	NSW	12
RAMCHEN Lev	Melbourne Grammar School	VIC	12
SLY Allan	Radford College	ACT	12
TONG Kester	Lake Ginninderra College	ACT	12
LU Yi-Hua (Sam)	Lake Ginninderra College	ACT	11
McNAMARA Peter	Hale School	WA	11
CHAN David	Sydney Grammar School	NSW	10
SEWELL Thomas	James Ruse Agricultural High School	NSW	12
NEW Elizabeth	James Ruse Agricultural High School	NSW	11
VUKCEVIC Damjan	Caulfield Grammar School	VIC	12
CHAN Vincent	Sydney Boys High School	NSW	12
THAI Duc	Rostrevor College	SA	11
MATHEWS Peter	Prince Alfred College	SA	11

Distinction

WONG Alex	James Ruse Agricultural High School	NSW	10
KULIK Yakov	Home Schooled	NSW	10
KWOK Andrew	University High School	VIC	11
MILLER Hugh	Barker College	NSW	11
GILBOURD Daniel	Sydney Boys High School	NSW	12
SLATYER Tracy	Narrabundah College	ACT	11
TAN Lit Hau	James Ruse Agricultural High School	NSW	11
HORNIBROOK Ken	Sydney Grammar School	NSW	11
RAMCHEN Bobbi	Melbourne Girls Grammar School	VIC	9
TSUI Chi Yuen	Bossley Park High School	NSW	12
ZHAO Yiying	Penleigh and Essendon Grammar School	VIC	10
WHITE Gareth	Hurlstone Agricultural High School	NSW	10

SHEPPARD Anna	Marbury School	SA	12
PHILLIPS Anthony	Hale School	WA	11
McDOWALL Jay	St Patrick's College	QLD	12
POPE Graeme	Merewether High School	NSW	11
RAO Gautam	Mentone Grammar School	VIC	10
SUEN Peter	North Sydney Boys High School	NSW	10
ROBERTS Ben	Toowoomba Grammar School	QLD	11
SHAW David	Radford College	ACT	12
SHERIDAN Nicholas	Scotch College	VIC	10
LOBO Erwin	Sefton High School	NSW	12
YAO Jialin	Cleeland Secondary College	VIC	11
CHUNG Hsiang	James Ruse Agricultural High School	NSW	12

Australian Team to participate in 2000 IMO at Taejon, South Korea

Geoffrey Chu, Thomas Xia, Allan Sly, Peter McNamara, David Chan, Thomas Sewell.

2001

Problems

PAPER 1 Tuesday 13 February

1. Let $L(n)$ be the least common multiple of $1, 2, \ldots, n$. Determine all pairs (p, q) of prime numbers such that $q = p+2$ and $L(q) > qL(p)$.

2. Let ABC be an isosceles triangle, with $AC = BC$. Let P, Q, R be points on AB, BC and AC, respectively, such that PQ is parallel to AC and PR is parallel to BC. Further, let O be the circumcentre of ABC.

Prove that the quadrilateral $CROQ$ is cyclic.

3. A town has c celebrities and m magazines. One week, each celebrity was mentioned in an odd number of magazines and each magazine mentioned an odd number of celebrities.

(a) Prove that m and c are either both even or both odd.

(b) In how many different ways can that mentioning of celebrities happen? Express this number in terms of c and m.

4. Prove that the polynomial $4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$ has no real root.

PAPER 2 Wednesday 14 February

5. Determine all functions f, defined for all real numbers and taking real numbers as values, for which

$$f((x-y)^2) = x^2 - 2yf(x) + (f(y))^2.$$

6. Let ABC be a triangle with $AC > BC$. On the circumcircle of the triangle ABC, let D be the midpoint of arc AB that contains C. Let E be the point on AC such that DE and AC are perpendicular.

Prove that $AE = EC + CB$.

7. Prove that there are no four positive integers w, x, y, z such that

$$\begin{aligned} x^2 &= 10w - 1 \\ y^2 &= 13w - 1 \\ z^2 &= 85w - 1. \end{aligned}$$

8. The country of Senso Unico has an airline whose flight routes are arranged as follows:

(i) whenever there is a direct route from city A to city B, then there is no direct route from city B to city A;

(ii) there is a route out of every city in Senso Unico;

(iii) whenever there is a direct route from city A to city C, then there is a city B such that there is a direct route from A to B and a direct route from B to C.

The population of Senso Unico is very proud to have the smallest possible number of cities that allows such an arrangement.

Determine this number.

Solution 1

Alternative 1 (Lit Hau Tan, James Ruse Agricultural High School, Sydney; also found by many others.)

Since $q = p + 2$ is prime, we immediately see $L(q) = qL(p + 1)$. We also note that $p = 2$ is not a solution, so that p is odd and thus $p + 1$ is even. Write $p + 1 = 2^m n$ with n odd. If $n > 1$ then $2^m < p$ and $n < p$ and $\gcd(2^m, n) = 1$ so that $L(p + 1) = L(p)$. Thus $L(q) = qL(p)$ which does not satisfy the inequality. On the other hand if $n = 1$ then we have three consecutive numbers $p, 2^m, q$ one of which must be divisible by 3. Since $3 \not| 2^m$ and both p and q are prime, we see that either $p = 3$ or $q = 3$. The latter does not yield a solution, while the former leads to the pair $(p, q) = (3, 5)$ which one can easily verify is a solution.

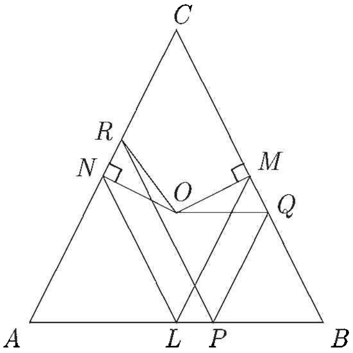
Alternative 2 (Stewart Wilcox, North Sydney Boys High School, NSW; also found by others.)

As in the first alternative we see p is odd and that $L(q) = qL(p+1)$. We thus require $L(p + 1) > L(p)$. Let the prime decomposition of $p + 1$ be $p + 1 = \prod_{i=1}^{k} p_i^{\alpha_i}$. If $k > 1$ then $p_i^{\alpha_i} \leq p$ for each i so that $p + 1|L(p)$ and so $L(p + 1) = L(p)$ which is no good. If $k = 0$ then $p + 1 = 1$ which is not a solution. If $k = 1$ then since $p + 1$ is even, we have $p + 1 = 2^\alpha$. If $\alpha = 2\beta$ is even, then $p = 2^{2\beta} - 1 = 4^\beta - 1 \equiv 1^\beta - 1 \equiv 0 \pmod 3$. If $\alpha = 2\beta + 1$ is odd, then $q = 2^{2\beta+1} + 1 = 2.4^\beta + 1 \equiv 2.1^\beta + 1 \equiv 0 \pmod 3$. Hence $3|p$ or $3|q$. But since p and q are both prime, $p = 3$ or $q = 3$, so $p = 1$ or 3. Since 1 is not prime, the only solution is $p = 3$ and $q = 5$. In this case we have $L(3) = 6$ and $L(5) = 60$, so $L(5) > 5L(3)$.

Solution 2

Alternative 1 (Charles Qin, Sydney Boys High School, NSW; Elena Kelareva, Elizabeth College, TAS; Tracy Slatyer, Narrabundah College, ACT; Gareth White, Hurlstone Agricultural High School, NSW)

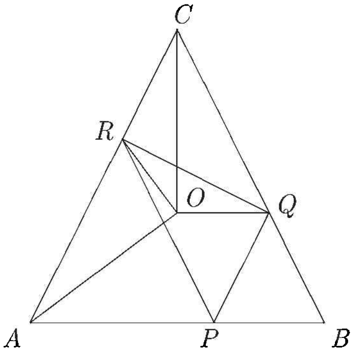
Let L, M and N be the midpoints of AB, BC and AC respectively. Since AB, BC and CA are all chords of the circle with centre O, the perpendiculars at L, M and N all meet at O. Furthermore the quadrilateral $CMON$ is cyclic. We can assume without loss of generality that P lies in the segment LB. Thus R lies in the segment CN and Q lies in the segment BM.



Now if we could show that $\triangle RON \equiv \triangle QOM$, then it would follow that $\angle RON = \angle QOM$ which quickly implies that $\angle ROQ + \angle RCQ = \angle NOM + \angle NCM = 180°$ from which it follows that $CROQ$ is a cyclic quadtrilateral.

Now we know triangles ACB, ARP and ANL are all similar isosceles triangles. Thus $CQ = RP = RA$ also $CM = NL = NA$. Hence $CQ - CM = RA - NA$, yielding $RN = MQ$. We also know that $ON = OM$ and that $\angle RNO = \angle QMO = 90°$. Thus triangles RON and QOM are congruent.

Alternative 2 (Brian Leung, Melbourne Grammar School, VIC. Similar solutions were also found by Peter McNamara, Hale School, WA; Bobbi Ramchen, Melbourne Girls' Grammar School, VIC; Yiying Zhao, Penleigh and Essondon Grammar School, VIC.)



We join AO and CO and notice that $\angle QCO = \angle OCA = \angle OAR$. Also $CQ = PR = AR$ and $CO = AO$. Thus $\triangle OCQ \equiv \triangle OAR$ (SAS). Hence $\angle CRO = 180° - \angle ARO = 180° - \angle CQO$. Thus $CROQ$ is cyclic.

Alternative 3 (Anthony Philips, Hale School, WA; Nicholas Luo, Melbourne Grammar School, VIC)

As in *Alternative 2* we establish $\triangle OCQ \equiv \triangle OAR$ (see the diagram in *Alternative 2*).
Thus $\angle AOC = \angle AOR + \angle COR = \angle COQ + \angle COR = \angle ROQ$.
Also $\dfrac{OR}{OQ} = 1 = \dfrac{OA}{OC}$. Thus $\triangle AOC \simeq \triangle ROQ$.
Hence $\angle RQO = \angle ACO = \angle RCO$. This establishes the concylicity of $CROQ$.

Solution 3

It is useful to think of this question pictorially. Set up a rectangular chessboard with m rows and c columns. If celebrity i is mentioned in magazine j, we colour the square in the ith column and the jth row red. The problem is equivalent to determining in how many ways it is possible to have an odd number of red squares in each row and each column.

Another interpretation is as follows: Construct a graph with c blue vertices representing celebrities and m red vertices representing magazines. We join a blue vertex to a red vertex by an edge if the corresponding celebrity is mentioned in the corresponding magazine. The problem is now equivalent to determining in how many ways it is possible to have such a bipartite graph (i.e. no two vertices of the same colour are joined by an edge) such that all vertices have odd degree.

Alternative 1 (Peter McNamara, Hale School, WA; Stewart Wilcox, North Sydney Boys High School, NSW)

Part (a): With the chessboard interpretation of the problem as discussed above, let a_i be the number of red squares in column i and b_i the number of red squares in row i. We are given that $a_i \equiv b_i \equiv 1 \pmod 2$. We now count the number of red squares in two ways - by columns then by rows. This yields $\sum_{i=1}^{c} a_i = \sum_{i=1}^{m} b_i$. Considering both sides modulo 2 yields $c \equiv m \pmod 2$. Thus m and c are either both even or both odd.

Part (b): Consider the subset of the board containing only columns 1 to $c-1$ and rows 1 to $m-1$. There are $(m-1)(c-1)$ squares in this sub-board and thus $2^{(m-1)(c-1)}$ possible colourings if there were no restrictions on the colouring. (Each square has two choices - coloured or not coloured.) Once any such colouring is considered we show how to uniquely extend such a colouring of the sub-board to a colouring of the whole board such that each row and column contains an odd number of red squares.

If row i $(1 \le i \le m-1)$ already has an odd number of squares then leave the last square uncoloured. If it has an even number of squares then colour the last square. Perform the analogous process for columns 1 to $c-1$. Thus we have uniquely extended the colouring to every square except the square in the mth row and cth column. This final square is now coloured/not coloured accordingly so as to ensure that the total number of coloured squares in column c is odd. This colours the whole board. Every row and every column has an odd number of coloured squares in it except perhaps the mth row. We now show that the mth row has an odd number of coloured squares. We have $a_i \equiv 1 \pmod 2$ for $1 \le i \le c$. We also have $b_i \equiv 1 \pmod 2$ for $1 \le i \le m-1$. As before we have $\sum_{i=1}^{c} a_i = \sum_{i=1}^{m} b_i$. Taking this equation modulo 2 yields $c \equiv m - 1 + b_m \pmod 2$. But recall also that $c \equiv m \pmod 2$ from part (a). Thus b_m is odd as required and the answer is $2^{(m-1)(c-1)}$.

Alternative 2 (Anthony Phillips, Hale School, WA. Similar solutions also found by Tracy Slatyer, Narrabundah College, ACT; Bobbi Ramchen, Melbourne Girls Grammar School, VIC.)

Part (a): With the graph theory interpretation as above, we know all $c+m$ vertex degrees v_i, are odd. But the number of edges is $\frac{1}{2}\sum_{i=1}^{c+m} v_i$, which is an integer. Thus $\sum_{i=1}^{c+m} v_i$ is even. Since each v_i is odd, we conclude that $c+m$ is even and thus c and m are of the same parity.

Part (b): We connect the first $c-1$ blue vertices to red vertices such that each of the first $c-1$ blue vertices has odd degree. We now show that we can uniquely complete the bipartite graph so that all vertices have odd degree by joining some red vertices the cth blue vertex. We do this as follows. If a red vertex has even degree, connect it to the cth blue vertex. If it has odd degree, do not connect it to the cth blue vertex. Thus all vertices now have odd degree except perhaps the cth blue vertex. However since the sum of the degrees of vertices in a graph must be even and we have an even number $c+m$ of vertices from part (a) we see that this final vertex must also have odd degree.

All that remains to do now is to enumerate the number of ways in which we can connect the first $c-1$ blue vertices to red vertices such that each of the $c-1$ blue vertices have odd degree. Consider one such blue vertex. Either we can connect it to $1, 3, 5, \ldots$ vertices. The number of ways for this one vertex is thus $\binom{m}{1} + \binom{m}{3} + \binom{m}{5} + \cdots = 2^{m-1}$. (Note that this identity can be established in many ways. One such way is to add together the two binomial expansions of $(1+1)^m$ and $(1-1)^m$.) However there are $c-1$ blue vertices all independently with 2^{m-1} possibilities. Multiplying all these together yields the answer $2^{(m-1)(c-1)}$.

Solution 4

Alternative 1 (Peter McNamara, Hale School, WA; David Chan, Sydney Grammar School, NSW; Maiyuran Arumugam, James Ruse Agricultural High School, NSW; Bobbi Ramchen, Melbourne Girls Grammar School, VIC)

The key to this problem is to express

$$p(x) = 4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$$

as a sum of squares. The $-2x^7 + x^6$ term suggests that

$$x^8 - 2x^7 + x^6 = (x^3(x-1))^2$$

might be one of those squares. Indeed with a little experiment, one can come up with

$$p(x) = 3\left(x^4 - \frac{1}{2}\right)^2 + \left(x^3(x-1)\right)^2 + \left(x - \frac{1}{2}\right)^2.$$

Thus $p(x)$ being a sum of squares is non-negative, furthermore $p(x) = 0$ only if all the squares are zero. However this is impossible because it simultaneously requires that $x = \pm\sqrt[4]{\frac{1}{2}}$ and $x = \frac{1}{2}$. Thus $p(x)$ has no real root.

Alternative 2 (Elizabeth New, James Ruse Agricultural High School, NSW)

After some experiment we find

$$p(x) = 2x^6(x - \frac{1}{2})^2 + \frac{1}{2}x^2(x^2-1)^2 + 2(x^4 - \frac{1}{2})^2 + \frac{1}{2}(x-1)^2.$$

Since a square is non-negative, $p(x) = 0$ only if all four squares are zero. In particular we simultaneously require $x = \pm\sqrt[4]{\frac{1}{2}}$ and $x = 1$ which is impossible. Thus $p(x)$ has no real root.

Solution 5

Alternative 1 (Peter McNamara, Hale School, WA)

Setting $x = y = 0$ yields $f(0) = (f(0))^2$. Thus $f(0) = 0$ or $f(0) = 1$.

Case 1: $f(0) = 0$. In this case setting $x = y$ yields:

$$\begin{aligned} f(0) &= x^2 - 2xf(x) + (f(x))^2 \\ 0 &= (x - f(x))^2 \\ 0 &= x - f(x) \\ f(x) &= x \text{ for all real } x. \end{aligned}$$

We now check that this is a solution. Indeed substitution yields:

$$\begin{aligned}\text{LHS} &= (x-y)^2\\ &= x^2-2xy+y^2\\ \text{RHS} &= x^2-2yf(x)+(f(y))^2\\ &= x^2-2xy+y^2\\ &= \text{LHS}.\end{aligned}$$

Case 2: $f(0)=1$. Setting $y=0$ yields:

$$f(x^2) = x^2+1$$

thus $f(t)=t+1$ for all $t\geq 0$. Thus if t is any positive real number we have:

$$\begin{aligned}f((x-t)^2) &= x^2-2tf(x)+(f(t))^2\\ (x-t)^2+1 &= x^2-2tf(x)+(t+1)^2 \text{ as } (x-t)^2, t\geq 0.\\ x^2-2xt+t^2+1 &= x^2-2tf(x)+t^2+2t+1\\ 2tf(x) &= 2xt+2t\\ f(x) &= x+1 \text{ for all real } x \text{ as } t\neq 0.\end{aligned}$$

We now check that this is a solution.

$$\begin{aligned}\text{LHS} &= (x-y)^2+1\\ &= x^2-2xy+y^2+1\\ \text{RHS} &= x^2-2y(x+1)+(y+1)^2\\ &= x^2-2xy-2y+y^2+2y+1\\ &= x^2-2xy+y^2+1\\ &= \text{LHS}.\end{aligned}$$

Thus we have two solutions: $f(x)=x$ and $f(x)=x+1$.

Alternative 2 (Graeme Pope, Merewether High School, NSW)
As in the first alternative, we establish the two cases $f(0)=0$ or $f(0)=1$.
Case 1: $f(0)=0$. Setting $y=0$ yields: $f(x^2)=x^2$. Thus $f(t)=t$ for all $t\geq 0$. We now set $x=0$ to obtain $(f(y))^2=f(y^2)=y^2$. Now set $x=y$ to obtain:

$$\begin{aligned}0 &= x^2-2xf(x)+(f(x))^2\\ &= x^2-2xf(x)+x^2\\ 2xf(x) &= 2x^2\\ f(x) &= x \text{ for } x\neq 0.\end{aligned}$$

This together with $f(0) = 0$ yields $f(x) = x$ for all real x. As in the first alternative we can verify this is a solution.

Case 2: $f(0) = 1$. Setting $y = 0$ yields $f(x^2) = x^2 + 1$. Thus $f(t) = t + 1$ for all $t \geq 0$. We now set $x = 0$ to obtain $f(y^2) = -2y + (f(y))^2$. Thus $(f(y))^2 = 2y + y^2 + 1$. Now set $y = x$ to obtain:

$$\begin{aligned} 1 &= x^2 - 2xf(x) + (f(x))^2 \\ &= x^2 - 2xf(x) + 2x + x^2 + 1 \\ 2xf(x) &= 2x^2 + 2x \\ f(x) &= x + 1 \text{ for } x \neq 0. \end{aligned}$$

This together with $f(0) = 1$ yields $f(x) = x + 1$ for all real x. As in the first alternative we can verify this is a solution.

Alternative 3 (Stewart Wilcox, North Sydney Boys' High School, NSW; Lit Hau Tan, James Ruse Agricultural High School, NSW.)

As in *Alternative 1* we can establish the two cases $f(0) = 0$ or $f(0) = 1$.

Case 1: $f(0) = 0$. Setting $y = 0$ yields $f(x^2) = x^2$ so that $f(t) = t$ for $t \geq 0$. Now let x be any real number. Since $(x-1)^2 \geq 0$, we have:

$$\begin{aligned} (x-1)^2 &= f((x-1)^2) \\ &= x^2 - 2f(x) + (f(1))^2 \\ x^2 - 2x + 1 &= x^2 - 2f(x) + 1 \\ f(x) &= x \end{aligned}$$

As in *Alternative 1* we can check that $f(x) = x$ is a solution.

Case 2: $f(0) = 1$. Setting $y = 0$ yields $f(x^2) = x^2 + 1$ so that $f(t) = t + 1$ for all $t \geq 0$. Now let x be any real number. We have:

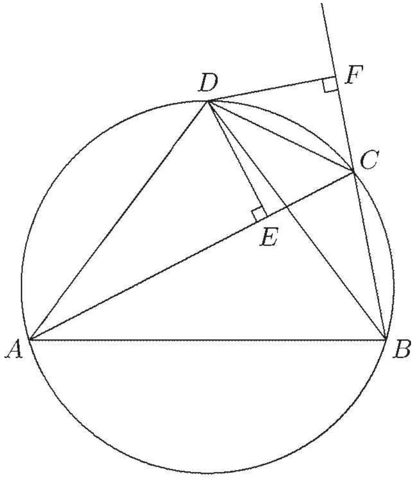
$$\begin{aligned} (x-1)^2 + 1 &= f((x-1)^2) \\ &= x^2 - 2f(x) + (f(1))^2 \\ x^2 - 2x + 2 &= x^2 - 2f(x) + 4 \\ f(x) &= x + 1 \end{aligned}$$

As in the first alternative we can check that $f(x) = x + 1$ is a solution.

Solution 6

Comment: The question itself suggests an approach using reflections or the marking off of equal lengths. A number of attacks along these lines are successful.

Alternative 1 (Francis Connon, Scotch College, VIC; Peter McNamara, Hale School, WA; also found by some others.)



Construct F on BC extended so that $DF \perp BC$. We see that $\angle FCD = \angle DAB = \angle DBA = \angle DCA = \angle ECD$.

Also $\angle DFC = \angle DEC = 90°$ and DC is common. Hence $\triangle DCF \equiv \triangle DCE$ (AAS). Hence $EC = FC$ and $DE = DF$.

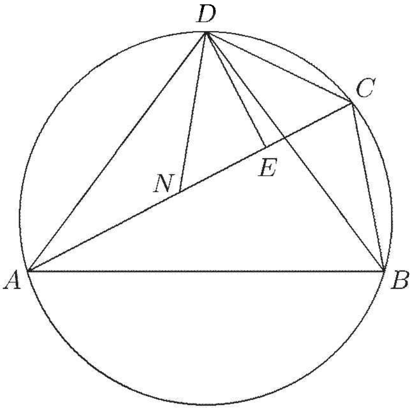
Since $DA = DB$ and $\angle DEA = \angle DFB = 90°$, we have $\triangle ADE \equiv \triangle BDF$.

Thus

$$AE = FB = FC + CB = EC + CB$$

as required.

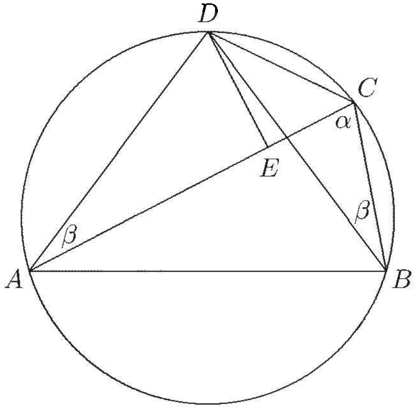
Alternative 2 (Bobbi Ramchen, Melbourne Girls Grammar School, VIC)



Construct the point N on AC such that $AN = BC$. Note that N lies on the interior of the segment AC because $AC > BC$. We now have that $\angle DAN = \angle DAC = \angle DBC$. Also $AD = BD$ and $AN = BC$. Thus $\triangle DAN \equiv \triangle DBC$ (SAS). Thus $ND = CD$. Hence $\triangle NDC$ is isosceles with DE being its altitude dropped to its base NC. Thus DE bisects NC and $EN = EC$. Hence $AE = EN + AN = EC + CB$ as required.

Comment: A very similar solution was found by Brian Leung, Melbourne Grammar School, VIC, who constructed the point N on AC such that $EN = EC$. He then proceeded to show that $AN = BC$ by showing that $\triangle DEN \equiv \triangle DEC$ and then $\triangle DAN \equiv \triangle DBC$.

Alternative 3 (Trigonometric solution by Peter Suen, North Sydney Boys High School, NSW.)



Let $\angle CAD = \beta$ and $\angle BCA = \alpha$, then also $\angle CBD = \beta$ and $\angle BDA = \alpha$. Since D is the midpoint of the arc AB, it follows that $AD = BD$ and that

$\angle DAB = \angle DBA = 90^\circ - \frac{1}{2}\alpha$. Thus also $\angle ACD = \angle ABD = 90^\circ - \frac{1}{2}\alpha$ and $\angle BAC = 90^\circ - \frac{1}{2}\alpha - \beta$. From the angle sum in triangle DEC we have that $\angle EDC = \frac{1}{2}\alpha$.

We now calculate everything in terms of DE and the angles α and β. The sine rule is used extensively. We have $AE = \dfrac{DE}{\tan\beta}$. Sine rule to triangle BCD yields:

$$\frac{BC}{\sin(90^\circ - \frac{1}{2}\alpha - \beta)} = \frac{CD}{\sin\beta}.$$

Hence

$$\begin{aligned} BC &= \frac{CD\sin(90^\circ - \frac{1}{2}\alpha - \beta)}{\sin\beta} \\ &= \frac{DE\sin(90^\circ - \frac{1}{2}\alpha - \beta)}{\sin\beta\sin(90^\circ - \frac{1}{2}\alpha)} \quad \left(\text{since } CD = \frac{DE}{\sin(90^\circ - \frac{1}{2}\alpha)}.\right) \end{aligned}$$

Hence now

$$\begin{aligned} EC + BC &= DE\tan\frac{1}{2}\alpha + DE\frac{\sin(90^\circ - \frac{1}{2}\alpha - \beta)}{\sin\beta\sin(90^\circ - \frac{1}{2}\alpha)} \\ &= DE\left(\tan\frac{1}{2}\alpha + \frac{\cos(\frac{1}{2}\alpha + \beta)}{\sin\beta\cos\frac{1}{2}\alpha}\right) \\ &= DE\left(\tan\frac{1}{2}\alpha + \frac{\cos\frac{1}{2}\alpha\cos\beta - \sin\frac{1}{2}\alpha\sin\beta}{\sin\beta\cos\frac{1}{2}\alpha}\right) \\ &= DE\left(\tan\frac{1}{2}\alpha + \frac{\cos\beta}{\sin\beta} - \frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha}\right) \\ &= \frac{DE}{\tan\beta} \\ &= AE \text{ as required.} \end{aligned}$$

Solution 7

Alternative 1 (David Chan, Sydney Grammar School, NSW)

We work modulo 16. Assume such positive integers can be found. Note that squares (mod 16) can only be 0, 1, 4 or 9. If

$$w \equiv 0, 2, 3, 4, 6, 7, 8, 10, 11, 12, 14, 15 \pmod{16}$$

then

$$x^2 \equiv 15, 3, 13, 7, 11, 5, 15, 3, 13, 7, 11, 5 \pmod{16}$$

respectively, which is impossible.
If $w \equiv 1, 13 \pmod{16}$ then $y^2 \equiv 12, 8 \pmod{16}$ respectively, which is impossible. Finally if $w \equiv 5, 9 \pmod{16}$ then $z^2 \equiv 8, 12 \pmod{16}$ respectively which is also impossible. Thus no such w can be found.

Alternative 2 (Maiyuran Arumugam, James Ruse Agricultural High School, NSW)

Assuming such integers exist, we eliminate w from the equations as follows:

$$\begin{aligned} z^2 &= 85w - 1 \\ &= 5(13w - 1) + 2(10w - 1) + 6 \\ &= 5y^2 + 2x^2 + 6 \end{aligned}$$

Consider the right-hand side of the final equation modulo 16. We know that squares (mod 16) can only be in the set $\{0, 1, 4, 9\}$. Thus

$$5y^2 \pmod{16} \in \{0, 5, 4, 13\},$$

and

$$2x^2 \pmod{16} \in \{0, 2, 8\}.$$

Thus considering all possible combinations yields that

$$z^2 = 5y^2 + 2x^2 + 6 \pmod{16} \in \{2, 3, 5, 6, 8, 10, 11, 12, 13, 14\}$$

which is impossible. Thus no such integers can be found.

Alternative 3 (Yiying Zhao, Penleigh and Essendon Grammar School, VIC)

We note that x^2, y^2 and z^2 modulo 16 can only be members of the set $\{0, 1, 4, 9\}$ and we thus try to find all possible $w \pmod{16}$. The first equation yields:

$$\begin{aligned} 10w - 1 &\equiv 0, 1, 4, 9 \pmod{16} \\ 10w &\equiv 1, 2, 5, 10 \pmod{16} \\ 10w &\equiv 2, 10 \pmod{16} \text{ (as } 10w \text{ is even)} \\ 10w &\equiv 50, 10 \pmod{16} \\ 5w &\equiv 25, 5 \pmod{8} \\ w &\equiv 5, 1 \pmod{8} \\ w &\equiv 1, 5, 9, 13 \pmod{16} \end{aligned}$$

The second equation yields:

$$\begin{aligned} 13w - 1 &\equiv 0, 1, 4, 9 \pmod{16} \\ 13w &\equiv 1, 2, 5, 10 \pmod{16} \\ 13w &\equiv 65, 130, 117, 26 \pmod{16} \\ w &\equiv 5, 10, 9, 2 \pmod{16} \end{aligned}$$

The third equation yields:

$$\begin{aligned} 85w-1 &\equiv 0,1,4,9 \pmod{16} \\ 5w &\equiv 1,2,5,10 \pmod{16} \\ 5w &\equiv 65,50,5,10 \pmod{16} \\ w &\equiv 13,10,1,2 \pmod{16} \end{aligned}$$

However there is obviously no integer w which can fit all three conditions.

Alternative 4 (Peter McNamara, Hale School, WA)

We have that $x^2 = 10w-1$ is odd so that x is odd. Thus $x^2 \equiv 1 \pmod{4}$. Thus we derive that $w \equiv 1 \pmod{2}$. So that x and w are both odd. Thus y^2 and z^2 are even. Hence we see that both y and z are even. As y is even we have:

$$\begin{aligned} y^2 &\equiv 0,4 \pmod{16} \\ 13w &\equiv 1,5 \pmod{16} \\ 65w &\equiv 5,25 \pmod{16} \\ w &\equiv 5,9 \pmod{16} \end{aligned}$$

As z is even we have:

$$\begin{aligned} z^2 &\equiv 0,4 \pmod{16} \\ 85w &\equiv 1,5 \pmod{16} \\ 5w &\equiv 1,5 \pmod{16} \\ 65w &\equiv 13,65 \pmod{16} \\ w &\equiv 13,1 \pmod{16} \end{aligned}$$

However there is no integer w which can fit these two conditions.

Alternative 5 (Problems Committee)

As in the fourth alternative we conclude that x is odd and y and z are even. Now we look for a combination of the three equations which eliminates w terms and constant terms. We find that

$$24x^2 - 25y^2 + z^2 = 0.$$

Since y and z are even, there are integers u and v such that $y = 2u$ and $z = 2v$.

Thus

$$24x^2 - 100u^2 + 4v^2 = 0$$

$$6x^2 = 25u^2 - v^2.$$

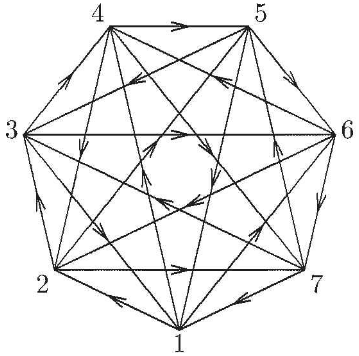
It follows that u and v are of the same parity so that the right-hand side is divisible by 4. Thus x is even. A contradiction.

Solution 8

Alternative 1 (Anthony Phillips, Hale School, WA. Similar solutions were found by Ross Atkins, Pembroke School, SA; Yiying Zhao, Penleigh and Essendon Grammar School, VIC, and a few others.)

Consider a city A. By (ii) there is a direct route out of A to C say. By (iii) there is a city B say, with direct routes from A to B and from B to C. However since there is a direct route from A to B now, then by (iii) again there is a city D say such that there are direct routes from A to D and from D to B. Furthermore $D \neq C$ as otherwise we would have direct routes from C to B and from B to C contradicting (i). Thus any given city has at least three distinct direct routes out of it. Assuming now that we have n cities, we have shown that the number of flight routes is at least $3n$. However this number by (i) cannot be greater than the number of pairs of cities. Thus $3n \leq \frac{1}{2}n(n-1)$, and so $n \geq 7$.

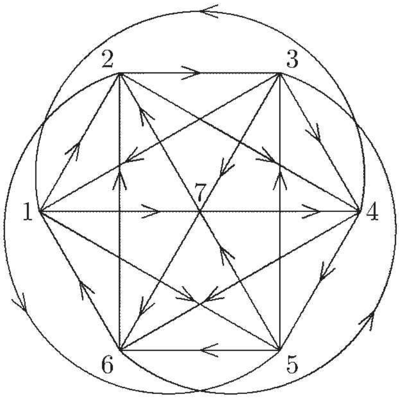
We now show that $n = 7$ is possible. Number the towns from 1 to 7. Have town 1 connected by direct flights to towns 2, 4 and 6 and receiving flights from towns 3, 5 and 7.



Repeat this arrangement cyclicly for each town so that town i is connected by direct flights to towns $i+1$, $i+3$ and $i+5$ and receives flights from towns $i+2$, $i+4$ and $i+6$ for each i. (Town numbers are taken modulo 7.) It easy to check that this arrangement satisfies the conditions.

Alternative 2 (Maiyuran Arumugam, James Ruse Agricultural High School, NSW; Hugh Miller, Barker College, NSW)

As in *Alternative 1*, we can establish that every city has at least three routes out of it. Assume now we have a direct route from C_1 to C_2. By (iii) there is a city C_3 such that we have direct routes from C_1 to C_3 and from C_3 to C_2. But since we now have a direct route from C_3 to C_2 then by (iii) again there is another city C_4 with such that we have direct routes from C_3 to C_4 and from C_4 to C_2. Note that $C_4 \neq C_1$ since otherwise we have direct routes from C_1 to C_3 and from C_3 to C_1, violating (i). Thus C_2 has at least three distinct routes into it. However we know that every city has at least three distinct routes out of it. Thus by (i) C_2 is connected to at least six other cities. Hence there are at least seven cities.



To show seven is achievable, label the cities 1 to 6 around the vertices of a regular hexagon with the 7th city in the middle. Have routes 1-2-3-4-5-6-1, 1-5-3-1, 2-4-6-2, 1-7-4-1, 3-7-6-3 and 5-7-2-5. It is easy to check that this arrangement satisfies the conditions.

Alternative 3 (Peter McNamara, Hale School, WA; Ken Hornibrook, Sydney Grammar School, NSW; Jensen Lai, Billanook College, VIC)

As in *Alternative 1*, we can establish that every city has at least three routes out of it. Hence if there are n cities then there are at least $3n$ different flight routes out of cities. But since the number of flights out equals the number of flights in, there are at least $3n$ routes into the n cities. Thus by the pigeonhole principle there exists a city with at least three flights into it. Such a city of course already has at least three flights out of it, hence by (i) it is connected to at least six different cities. Thus there are at least seven cities. See *Alternatives 1 and 2* for two of the many ways in which this number can be realised.

RESULTS

Number of Candidates: 121

Gold

McNAMARA Peter	Hale School	WA	12
CHAN David	Sydney Grammar School	NSW	11
PHILLIPS Anthony	Hale School	WA	12
ZHAO Yiying	Penleigh and Essendon Grammar School	VIC	11
ARUMUGAM Maiyuran	James Ruse Agricultural High School	NSW	12
RAMCHEN Bobbi	Melbourne Girls Grammar School	VIC	10
WILCOX Stewart	North Sydney Boys High School	NSW	11
MILLER Hugh	Barker College	NSW	12
NEW Elizabeth	James Ruse Agricultural High School	NSW	12
QIN Charles	Sydney Boys High School	NSW	10

Silver

LAI Jensen	Billanook College	VIC	10
LU Yi-Hua	Narrabundah College	ACT	12
WHITE Gareth	Hurlstone Agricultural High School	NSW	11
WONG Alex	James Ruse Agricultural High School	NSW	11
CHEN Zhihong	Melbourne High School	VIC	10
HALL Cameron	Cannon Hill Anglican College	QLD	12
SLATYER Tracy	Narrabundah College	ACT	12
TAN Lit Hau	James Ruse Agricultural High School	NSW	12
KWOK Andrew	University High School	VIC	11
SUPASITI Tharatorn	Scotch College	VIC	11
KELAREVA Elena	Elizabeth College	TAS	11
SHERIDAN Nicholas	Scotch College	VIC	11
HORNIBROOK Ken	Sydney Grammar School	NSW	12
ZHAO James	University High School	VIC	10
LUO Nicholas	Melbourne Grammar School	VIC	11
DEUTSCHER Nathan	Braemar College	VIC	10
MA George	Brisbane Grammar School	QLD	12

ATKINS Ross	Pembroke School	SA	10
POPE Graeme	Merewether High School	NSW	12
COLEMAN Patrick	St Peters College	SA	9
KHAN Kamil	St Michaels Grammar School	VIC	9
CONNON Francis	Scotch College	VIC	12
ROBERTS Ben	Toowoomba Grammar School	QLD	12

Australian Team to participate in 2001 IMO at Washington, USA

Peter McNamara, David Chan, Zhao Yiying, Bobbi Ramchen, Stewart Wilcox, Hugh Miller.

2002

Problems

PAPER 1 Tuesday 12 February

1. Let m and n be positive integers such that

$$2001m^2 + m = 2002n^2 + n$$

holds.

Prove that $m - n$ is a perfect square.

2. Determine all triples (u, v, w) of real numbers satisfying

(i) $u + v + w = 38$,

(ii) $uvw = 2002$,

(iii) $0 < u \leq 11$, $w \geq 14$.

3. A line through a vertex of a triangle is called a dividend if it cuts this triangle into two triangles with equal perimeters.

Let ABC be a triangle. Prove that the dividends of ABC are concurrent.

4. Determine the largest positive integer n for which there exists a set S with exactly n numbers such that

(i) each number in S is a positive integer not exceeding 2002;

(ii) if a and b are two (not necessarily different) numbers in S, then their product ab does not belong to S.

PAPER 2 Wednesday 13 February

5. Determine all real-valued functions f defined for all real numbers which satisfy

$$f(2002x - f(0)) = 2002x^2$$

for each real x.

6. Let $ABCD$ be a rectangle, let E be a point on BC and F a point on CD such that AEF is an equilateral triangle.

Prove that the area of triangle ECF equals the sum of the areas of triangle ABE and triangle AFD.

7. Let n and q be integers, $n \geq 5$, $2 \leq q \leq n$. Prove that $q - 1$ divides $\left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

(Note: $\lfloor x \rfloor$ is the largest integer not exceeding x.)

8. Let a, b, c be real numbers, and suppose that A, B, C, D are real numbers such that

$$(Ax + B)(Cx + D) = ax^2 + bx + c$$

for all real numbers x.

Prove that at least one of the numbers a, b, c is greater than or equal to $\frac{4}{9}(A + B)(C + D)$.

Solution 1

Alternative 1 (Ivan Guo, Sydney Boys' High School, NSW; Gareth White, Hurlstone Agricultural High School, NSW)

Since m and n are positive integers, it is clear that $m > n$. Let $m = n + k$ where k is a positive integer. The equation after simplification now becomes

$$n^2 - 4002nk - 2001k^2 - k = 0.$$

Viewing this as a quadratic in n we see that for n to be an integer, the discriminant must be a perfect square. Thus

$$\begin{aligned} D^2 &= (4002k)^2 + 4(2001k^2 + k) \\ \left(\tfrac{D}{2}\right)^2 &= k((2001^2 + 2001)k + 1). \end{aligned}$$

Since $\gcd(k, (2001^2{+}2001)k{+}1) = 1$ and their product is a perfect square, it follows that both k and $(2001^2 + 2001)k + 1$ are perfect squares. Thus $m - n = k$ is a perfect square.

Alternative 2 (Jennifer Wang, MLC School, NSW)

Using the notation of *Alternative 1* we establish

$$n^2 - 4002nk - 2001k^2 - k = 0.$$

Completing the square in n yields

$$(n - 2001k)^2 = k(2001 \cdot 2002k + 1).$$

If $k = 1$ then $m - n = k = 1$ is a perfect square and we are finished. If $k > 1$, let q be any prime divisor of k and let q^α be the highest power of q dividing k. Since $q \nmid 2001 \cdot 2002k + 1$ then q^α is the highest power of q dividing $(n - 2001k)^2$. Thus α is even. Since this is true for all primes dividing k we conclude that $k = m - n$ is a perfect square.

Alternative 3 (Stewart Wilcox, North Sydney Boys High School, NSW; Andrew Kwok, University High School, VIC; Zhihong Chen, Melbourne High School, VIC)

The given equation can be rewritten as

$$(m - n)(1 + 2001(m + n)) = n^2.$$

Now assume that $p | m - n$ and $p | 1 + 2001(m + n)$ for some prime p. Then since $m - n | n^2$ we have $p | n^2$ and thus $p | n$ since p is prime. Thus also $p | (m - n) + n$, i.e. $p | m$. So $p | 2001(m + n)$. But $p | 1 + 2001(m + n)$. Thus $p | 1$. A contradiction so our assumption is false. Thus $\gcd(m - n, 1 + 2001(m{+}n)) = 1$. However their product is a perfect square and they are both positive. Hence they must both be perfect squares. In particular $m - n$ is a perfect square.

Alternative 4 (Bobbi Ramchen, Melbourne Girls' Grammar School, VIC; Nicholas Sheridan, Scotch College, VIC)

We prove that $m-n=(\gcd(m,n))^2$. As in the third alternative we have

$$(m-n)(1+2001(m+n))=n^2.$$

Now let $\gcd(m,n)=d$ so we can write $m=da$ and $n=db$ where a, b are relatively prime positive integers. The equation now becomes

$$(a-b)(1+2001d(a+b))=db^2.$$

Now $\gcd(d,1+2001d(a+b))=1$. Thus $d|a-b$. However since $\gcd(a,b)=1$, we also have $\gcd(a-b,b)=1$ and $\gcd(a-b,b^2)=1$. Thus $a-b|d$. We conclude that $a-b=d$ and so $m-n=d(a-b)=d^2$ and we are finished.

Solution 2

Alternative 1 (Nicholas Sheridan, Scotch College, VIC)

Make the substitution $U=\frac{u}{11}$, $V=\frac{v}{13}$ and $W=\frac{w}{14}$. (This is motivated by noticing that $(u,v,w)=(11,13,14)$ is a solution. We now wish to show that $(U,V,W)=(1,1,1)$ is the only solution.)

The three given conditions now become

$$\begin{array}{lrcl}
(1) & 11U+13V+14W & = & 38, \\
(2) & UVW & = & 1, \\
(3) & 0<U\le 1 & , & W\ge 1.
\end{array}$$

From (2) and (3) we also have that $V=\frac{1}{UW}>0$. Thus U,V,W are all positive reals, so we can apply the weighted AM-GM inequality which tells us that:

$$\frac{11U+13V+14W}{11+13+14}\ge\left(U^{11}V^{13}W^{14}\right)^{\frac{1}{11+13+14}}.$$

Using (1) and (2) the above inequality can be rewritten as $U^{-3}V^{-1}\le 1$, i.e. $V\ge\frac{1}{U^3}$. Substituting this into (2) yields $1=UVW\ge U\cdot\frac{1}{U^3}\cdot W$. Thus $U^2\ge W\ge 1$. But (3) tells us that $1\ge U^2$. Thus the squeeze principle applies and we have $1=U^2=W$. Thus $U=W=1$ and also $V=\frac{1}{UW}=1$. Thus $(U,V,W)=(1,1,1)$, i.e. $(u,v,w)=(11,13,14)$ is the only solution to the given system.

Alternative 2 (Problems Committee)

Let

$$\begin{aligned} p(x) &= (x-11)(x-13)(x-14) = x^3 - 38x^2 + 479x - 2002, \\ q(x) &= (x-u)(x-v)(x-w). \end{aligned}$$

Then $q(x) = x^3 - 38x^2 + kx - 2002$ for some real number k. Put $d(x) = p(x) - q(x) = (479 - k)x$. As $q(u) = 0$, we have $d(u) = p(u) = (u - 11)(u-13)(u-14) \leq 0$ by (iii). Also $d(u) = (479-k)u$. Thus $479-k \leq 0$.

Similarly $d(w) = p(w) = (w-11)(w-13)(w-14) \geq 0$ by (iii). Also $d(w) = (479-k)w$. Thus $479 - k \geq 0$.

Hence we conclude that $k = 479$ and thus $p(x) = q(x)$. In particular these two polynomials have the same zeros. Hence $(u, v, w) = (11, 13, 14)$ is the only solution.

Alternative 3 (Angelo Di Pasquale, based on ideas of attempts of some students)

We prove that if conditions (i) and (iii) are met but $(u, v, w) \neq (11, 13, 14)$ then $uvw < 2002$.

Lemma: If $0 < a_0 < a_1 \leq b_1 < b_0$ are real numbers with $a_0 + b_0 = a_1 + b_1$ then $a_0 b_0 < a_1 b_1$. Indeed if

$$m = \frac{a_0 + b_0}{2} = \frac{a_1 + b_1}{2}$$

then

$$a_0 b_0 = m^2 - \left(\frac{a_0 - b_0}{2}\right)^2 < m^2 - \left(\frac{a_1 - b_1}{2}\right)^2 = a_1 b_1.$$

We now apply the lemma so as to squeeze u and w together until either $u = 11$ or $w = 14$. If it ends up after squeezing that $u = 11$ then $w \geq 14$ and so $v \leq 13$. Thus we can now apply the lemma so as to squeeze v and w together until $v = 13$ and $w = 14$. Similarly if after the initial squeezing of u and w it turns out that $w = 14$, then $u \leq 11$ and so $v \geq 13$. We now squeeze u and v together until $u = 11$ and $v = 13$.

To conclude the argument we now note that initially $(u, w) \neq (11, 14)$ and that according to the lemma, the squeezing operations performed could only increase the product uvw. At the end we obtained $(u, v, w) = (11, 13, 14)$ and $uvw = 2002$. Thus initially we must have had $uvw < 2002$ as claimed. Thus $(11, 13, 14)$ is the only solution.

Solution 3 (Andrew Kwok, University High School, VIC, and others)

Let the dividend from vertices A, B and C cut the opposite side in A', B' and C' respectively. As usual denote side lengths $BC = a$, $CA = b$ and $AB = c$. Let $BA' = m$ and $A'C = n$. Since AA' is a dividend, we have

$$\begin{aligned} c + m + AA' &= b + n + AA' \\ \text{i.e.} \qquad m - n &= b - c. \end{aligned}$$

We also have

$$m + n = a.$$

Adding yields $2m = a+b-c$. Subtracting yields $2n = a+c-b$. Dividing now yields

$$\frac{m}{n} = \frac{a+b-c}{a+c-b}.$$

We similarly obtain expressions for $\frac{CB'}{B'A}$ and $\frac{AC'}{C'B}$. We now compute the product to be

$$\begin{aligned} \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} &= \frac{a+b-c}{a+c-b} \cdot \frac{b+c-a}{b+a-c} \cdot \frac{c+a-b}{c+b-a} \\ &= 1. \end{aligned}$$

Thus by (the converse of) Ceva's Theorem AA', BB' and CC' are concurrent.

Solution 4

Alternative 1 (Yiying Zhao, Penleigh and Essendon Grammar School, VIC)

We claim that $S = \{45, 46, 47, \ldots, 2002\}$ is the largest set satisfying the conditions.

Firstly note that if $a, b \in S$ then $ab \geq 45^2 = 2025 > 2002$. Thus S satisfies the conditions and $n = |S| = 1958$.

Proof that $\max(n) = 1958$: Suppose to the contrary that there is another set T satisfying the conditions but $|T| > 1958$. Note that T contains a member e such that $e < 45$. We show that each member $e \in T$ with $e < 45$ disqualifies some other number, say $e' \geq 45$ ($e' \leq 2002$) from being in T and that all the e' are different.

Now if $7 \leq e \leq 44$ then condition (ii) disqualifies $e^2 \in T$. Note also that $e = 1$ is trivially disqualified. If $2 \in T$ then one of $48, 96 \notin T$. If $3 \in T$ then one of $50, 150 \notin T$. If $4 \in T$ then one of $47, 188 \notin T$. If $5 \in T$ then one of $46, 230 \notin T$. If $6 \in T$ then one of $45, 270 \notin T$.

Since the numbers $7^2, 8^2, \ldots, 44^2, 48, 96, 50, 150, 47, 188, 46, 230, 45, 270$ are all different we are finished.

Alternative 2 (Elena Kelareva, Elizabeth College, TAS)

Consider the 44 triples $\{44-x, 45+x, (44-x)(45+x)\}$ where $0 \leq x \leq 43$. Note that

$$(44-x)(45+x) = 1980 - x - x^2 < 2002.$$

From condition (ii) at least one member of each triple cannot be in S. Since all triples are disjoint we see that S must exclude at least 44 numbers. Thus S has at most $2002 - 44 = 1958$ members.

We still must show that $|S| = 1958$ is attainable. This can be done as in the first alternative.

Alternative 3 (Bobbi Ramchen, Melbourne Girls Grammar School, VIC)

Consider the following sets.
$A = \{1\}$, $B = \{2^1, 2^2, \ldots, 2^{10}\}$, $C = \{3^1, 3^2, \ldots 3^6\}$, $D = \{5^1, 5^2, 5^3, 5^4\}$, $E = \{6^1, 6^2, 6^3, 6^4\}$ and the sets $X_i = \{i, i^2\}$ where $7 \leq i \leq 44$ and i is not a power of $2, 3, 4, 5$ or 6. One can show that any valid set S must exclude at least 1 member from A, 5 members from B, 3 members from C, 2 members from D, 2 members from E and one member from each of the 31 sets X_i. Thus S can have at most $2002-(1+5+3+2+2+31) = 1958$ elements.

As in *Alternative 1* we show that $|S| = 1958$ is possible.

Solution 5 (Daniel Nadasi, Cranbrook School, NSW, and others)

Setting $x = \frac{f(0)}{2002}$ in the functional equation yields $f(0) = \frac{(f(0))^2}{2002}$. Thus $f(0) = 0$ or $f(0) = 2002$.

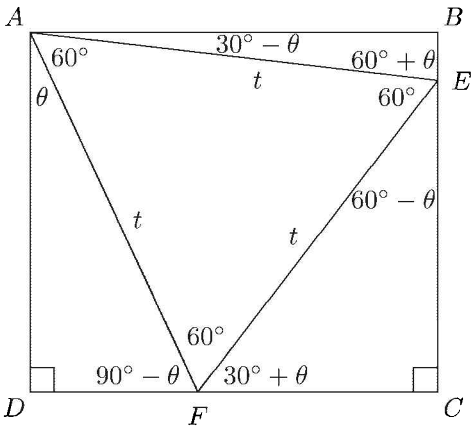
If $f(0) = 0$ then set $x = \frac{y}{2002}$ for any real number y. This immediately yields $f(y) = \frac{y^2}{2002}$.

If $f(0) = 2002$ then set $y = 2002x - 2002$ which is the same as $x = \frac{y+2002}{2002}$ for any real number y. This immediately yields $f(y) = \frac{(y+2002)^2}{2002}$.

Thus we have two potential solutions to the functional equation, namely $f(x) = \frac{x^2}{2002}$ and $f(x) = \frac{(x+2002)^2}{2002}$. It is still necessary to check whether or not they are indeed solutions. Routine substitution verifies this.

Solution 6

Alternative 1 (Virtually all successful attempts were by trigonometry. The following solution is by Yiying Zhao, Penleigh and Essendon Grammar School, VIC. A similar solution was found by Zhihong Chen, Melbourne High School, VIC.)



Let $/DAF = \theta$, then all other angles can be filled in as shown in the diagram. If the side length of the triangle is t, then

$$2 \cdot \text{Area}(ADF) = AD \cdot DF = t\sin\theta \cdot t\cos\theta = \frac{1}{2}t^2\sin 2\theta,$$

$$\begin{aligned} 2 \cdot \text{Area}(ABE) &= AB \cdot BE \\ &= t\sin(30^\circ - \theta) \cdot t\cos(30^\circ - \theta) \\ &= \frac{1}{2}t^2\sin(60^\circ - 2\theta), \end{aligned}$$

and

$$\begin{aligned} 2 \cdot \text{Area}(EFC) &= EC \cdot CF \\ &= t\sin(\theta + 30^\circ) \cdot t\cos(\theta + 30^\circ) \\ &= \frac{1}{2}t^2\sin(2\theta + 60^\circ). \end{aligned}$$

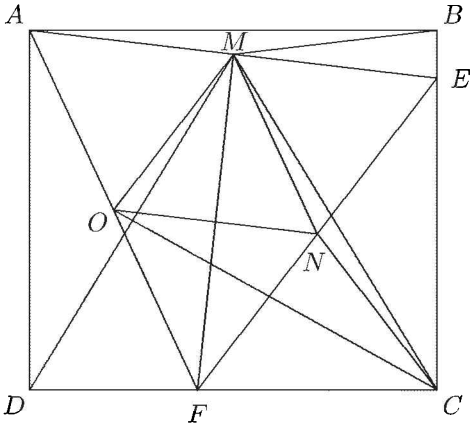
Thus it suffices to prove

$$\begin{aligned} \sin(2\theta + 60^\circ) &= \sin(60^\circ - 2\theta) + \sin 2\theta \\ \text{i.e.} \qquad \sin 2\theta &= \cos(30^\circ - 2\theta) - \cos(30^\circ + 2\theta). \end{aligned}$$

However from the relation $\cos A - \cos B = 2\sin(\frac{B+A}{2})\sin(\frac{B-A}{2})$ we see that the RHS of the last equation equals $2\sin 2\theta \sin 30^\circ =$ LHS as required.

N.B. Stewart Wilcox, North Sydney Boys High School, NSW, gave a similar proof showing that Area(ECF) = Area(AFD) – Area(ABE) if E is on CB extended and Area(ECF) = Area(ABE) – Area(AFD) if F is on CD extended.

Alternative 2 (A Euclidean solution provided by the Problems Committee)



Let M, N, O be the midpoints of AE, EF and FA respectively. Hence we conclude that triangle CMD is isosceles with $CM = DM$. Since $FM \perp AE$ we have that $FMEC$ is cyclic and so $\angle FCM = \angle FEM = 60^\circ$. Hence triangle DMC is in fact equilateral. We now see that triangles MNC and AMB are congruent. ($MC = DC = AB$ and $NC = NE = MN = AM = BM$.) Thus Area(ABE) = $2\cdot$Area(MNC). Similarly one shows that Area(ADF) = $2 \cdot$ Area(ONC). We also know that Area(FEC) = $2 \cdot$ Area(NEC).

Thus it suffices to show that Area(NEC) = Area(MNC) + Area(ONC). Yet these three triangles all have the same base, namely NC. Since EN and MO are opposite sides of a parallelogram, the sum of the distances from M and O to NC equals the distance from E to NC, and so the areas add as required.

Solution 7

Alternative 1 (Stewart Wilcox, North Sydney Boys High School, NSW)

Case 1: $q < n$. Then the product $(n-1)! = 1 \cdot 2 \cdot \ldots \cdot (n-1)$ contains both q and $q-1$ in its expansion. So $q(q-1)|(n-1)!$. In particular $(q-1)|\frac{(n-1)!}{q} = \left\lfloor \frac{(n-1)!}{q} \right\rfloor$.

Case 2: $q = n$, a prime number. Then by Wilson's Theorem we have

$$(n-1)! \equiv -1 \equiv n-1 \pmod{n}.$$

Also

$$(n-1)! \equiv 0 \equiv n-1 \pmod{n-1}.$$

Since $\gcd(n, n-1) = 1$ we can apply the Chinese Remainder Theorem to get

$$(n-1)! \equiv n-1 \pmod{n(n-1)}.$$

Thus $(n-1)! = kn(n-1) + n - 1$ for some integer k. Thus $\left\lfloor \frac{(n-1)!}{q} \right\rfloor = \left\lfloor k(n-1) + \frac{n-1}{n} \right\rfloor = k(n-1)$ which is divisible by $q - 1 = n - 1$.

Case 3: $q = n$, a composite number. It suffices to prove that $n(n-1)|(n-1)!$.

Let p be the largest prime factor of n and write $n = px$. Note that x is an integer greater than 1 and less than n. In fact since $x|n$ we cannot have $x = n-1$, so $x \le n-2$.

Case 3a: Now if p and x are distinct then they both occur at different places in the expansion of the product $(n-2)! = 1 \cdot 2 \cdot \ldots \cdot (n-2)$ so that $n = px|(n-2)!$ and thus $n(n-1)|(n-1)!$.

Case 3b: If $p = x$, then $n = p^2$. Since $n > 4$, we have $p > 2$ and thus $p^2 > 2p$. Note that $\gcd(2p, n) = p$ thus we cannot have $2p = n-1$, thus $2p \le n-2$. Hence p and $2p$ occur at different places in the expansion of the product $(n-2)! = 1 \cdot 2 \cdot \ldots \cdot (n-2)$ so that $2p^2|(n-2)!$ and thus $n(n-1)|(n-1)!$.

Alternative 2 (Variant by Andrew Kwok, University High School, VIC)

Case 1: $q < n$ is treated as in the first alternative.

Case 2: $q = n$, a prime number. From Wilson's Theorem we have $(n-1)! \equiv -1 \equiv n-1 \pmod{n}$. Hence $(n-2)! \equiv 1 \pmod{n}$. Hence $(n-2)! = yn + 1$ for some integer y. Thus

$$\begin{aligned} \left\lfloor \frac{(n-1)!}{n} \right\rfloor &= \left\lfloor \frac{(n-1)(yn+1)}{n} \right\rfloor \\ &= \left\lfloor ny - y + 1 - \frac{1}{n} \right\rfloor \\ &= (n-1)y. \end{aligned}$$

which is divisible by $n-1$.

Case 3: $q = n$, a composite number. Write $n = ab$, $(a, b > 1)$. We now have three further subcases.

Case 3a: $a \ne b$. This case can be treated as in case 3a in *Alternative 1.*

Case 3b: $a = b$, both prime numbers. This is case 3b in the *Alternative 1.*

Case 3c: $a = b$, both composite numbers. Let $a = b = cd$, $c, d \geq 2$. Then $n = c^2d^2$. Take $a_1 = c$ and $b_1 = cd^2$, then we have $a_1 < b_1$ and $n = a_1b_1$ which can be dealt with as in case 3a.

Alternative 3 (Variant by Elena Kelareva, Elizabeth College, TAS. A similar variant was completed by Gareth White, Hurlstone Agricultural High School, NSW.)

Case 1: $q < n$ is treated as in the first alternative.

Case 2: $q = n$, a prime number. Using Wilson's Theorem we quickly derive $(n-1)! - (n-1) \equiv 0 \pmod{n}$ $(*)$. Hence

$$\begin{aligned}\left\lfloor \frac{(n-1)!}{n} \right\rfloor &= \left\lfloor \frac{(n-1)! - (n-1)}{n} + \frac{n-1}{n} \right\rfloor \\ &= \frac{(n-1)! - (n-1)}{n} \\ &= \frac{(n-1)((n-2)! - 1)}{n}.\end{aligned}$$

Now we know that the last expression is an integer from $(*)$. Since $\gcd(n, n-1) = 1$ we have that $n | ((n-2)! - 1)$ and so the RHS of the last expression is divisible by $n - 1$.

Case 3: $q = n$, a composite number. We now have two further cases.

Case 3a: n is not a prime power. Then we can write $n = ab$ where $\gcd(a, b) = 1$, in particular $a \neq b$. We can then proceed as in case 3a of the second alternative.

Case 3b. n is a prime power. Write $n = p^b$ where p is prime and $b \geq 2$. Then in the expansion of $(n-2)!$ the factors p^{b-1} and $p^{b-1} + p$ occur thus $n = p^b | (n-2)!$ and so $n(n-1) | (n-1)!$ as required. The only reason this may fail is if $p^{b-1} + p \geq n - 1$. Now $p^{b-1} + p = n - 1$ is impossible because $p | n$. Thus we would have $p^{b-1} + p \geq n = p^b$. This would imply $1 \geq p^{b-2}(p-1)$, however this is only possible if $p = b = 2$ in which case $n = 4$ which has been disallowed.

Solution 8

Alternative 1 (Elena Kelareva, Elizabeth College, TAS)

Putting $x = 1$ yields $(A + B)(C + D) = a + b + c$. Thus we are required to prove that one of $a \geq \frac{4}{9}(a + b + c)$, $b \geq \frac{4}{9}(a + b + c)$ or $c \geq \frac{4}{9}(a + b + c)$ must be true. Suppose to the contrary that they are all false. Then we quickly obtain the three inequalities

$$5a < 4b + 4c \tag{1}$$

$$5b < 4a + 4c \tag{2}$$

$$5c < 4a + 4b. \tag{3}$$

We also note that the original quadratic has two real roots (namely $-\frac{B}{A}$ and $-\frac{D}{C}$) hence it must have positive discriminant. Thus

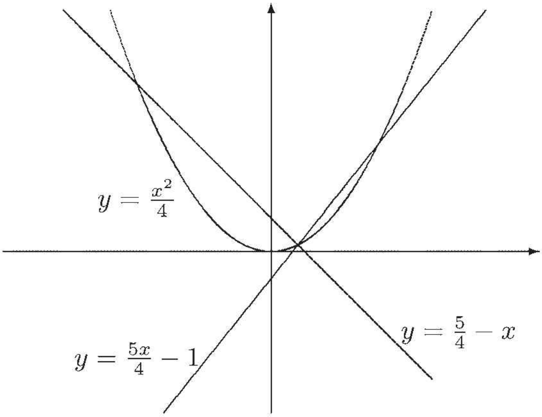
$$b^2 \geq 4ac. \tag{4}$$

Now adding (1), (2) and (3) yields $5(a+b+c) < 8(a+b+c)$ and thus $a+b+c > 0$. Adding (2) and (3) yields $c < 8a - b$. Hence $5a < 4b + 4c < 4b + 4(8a-b) = 32a$. Thus $a > 0$. Similarly $b, c > 0$.

We now set $X = \frac{b}{a}$ and $Y = \frac{c}{a}$ (note $X, Y > 0$). Since $a > 0$ we can safely substitute into (1), (2), (3) and (4) and clear denominators. We obtain the equivalent inequalities

$$\begin{aligned} Y &> \frac{5}{4} - X & (1)\\ Y &> \frac{5}{4}X - 1 & (2)\\ Y &< \frac{4}{5} + \frac{4}{5}X & (3)\\ Y &\leq \frac{1}{4}X^2. & (4)\end{aligned}$$

Combining (2) and (3) yields $\frac{5}{4}X - 1 < Y < \frac{4}{5} + \frac{4}{5}X$ from which we find $X < 4$. Also the graphs of $Y = \frac{1}{4}X^2$ and $Y = \frac{5}{4}X - 1$ intersect when $\frac{1}{4}X^2 = \frac{5}{4}X - 1$ which has solutions $X = 4$ and $X = 1$. Examining the graphs of (2) and (4) then shows that $X > 4$ or $X < 1$. However we already know $X < 4$. Hence $X < 1$.



Thus from (1) we have $Y > \frac{5}{4} - X > \frac{5}{4} - 1 = \frac{1}{4}$. However, $Y \leq \frac{1}{4}X^2 < \frac{1}{4}$ since $0 < X < 1$. This is a contradiction and so we are finished.

Alternative 2 (Based on the solution by Marina Pasika, St. Leonard's College, VIC)

Let $s = (A + B)(C + D)$. Suppose to the contrary that $a, b, c < \frac{4}{9}s$.

Case 1: $s \le 0$. Then adding yields $a + b + c < \frac{4}{3}s$. Note that putting $x = 1$ yields $s = a + b + c$. Thus $s < \frac{4}{3}s$. Thus $s > 0$. A contradiction.

Case 2: $s > 0$. We can assume without loss of generality that $s = 1$. (This is achieved by scaling the question so that we are really dealing with $(a', b', c') = (\frac{a}{s}, \frac{b}{s}, \frac{c}{s})$ and $(A', B', C', D') = (\frac{A}{\sqrt{s}}, \frac{B}{\sqrt{s}}, \frac{C}{\sqrt{s}}, \frac{D}{\sqrt{s}})$.)

Now if $a \le 0$, then $b + c = 1 - a \ge 1$ and thus $\max\{b, c\} \ge \frac{1}{2} > \frac{4}{9}$ and we are done. Similar results hold for b and c. Hence we have $0 < a, b, c < \frac{4}{9}$. Since the original quadratic has two real roots, its discriminant is positive. Thus we have $b^2 - 4ac \ge 0$. Hence the following inequality and equation hold:

$$\begin{aligned} ac &\le \frac{b^2}{4} \\ a + c &= 1 - b. \end{aligned}$$

Therefore a and c are solutions to the quadratic

$$y^2 - (1 - b)y + p = 0$$

where $p = ac \le \frac{b^2}{4}$. The roots of this quadratic are

$$y_{1,2} = \frac{1}{2}(1 - b \pm \sqrt{(1 - b)^2 - 4p}).$$

So the bigger one of these is

$$\begin{aligned} \frac{1 - b + \sqrt{(1 - b)^2 - 4p}}{2} &\ge \frac{1 - b + \sqrt{b^2 - 2b + 1 - 4 \cdot \frac{b^2}{4}}}{2} \\ &= \frac{1 - b + \sqrt{1 - 2b}}{2} \\ &> \frac{1 - \frac{4}{9} + \sqrt{1 - 2 \cdot \frac{4}{9}}}{2} \qquad \text{(since } b < \tfrac{4}{9}\text{)} \\ &= \frac{4}{9}. \end{aligned}$$

Therefore the larger of a and c is at least $\frac{4}{9}$ and the question is proven.

Alternative 3 (Jamie Simpson and Bolis Basit Saad)

As in the first alternative we arrive at the inequalities (1)-(4) as well as the facts that $a, b, c > 0$. From (2) and (4) we have $(\frac{4}{5}(a+c))^2 > b^2 \ge 4ac$. And so

$$\begin{aligned} 4a^2 - 17ac + 4c^2 &> 0 \\ (a - 4c)(4a - c) &> 0. \end{aligned}$$

This implies that either $a > 4c$ or $c > 4a$. However our inequalities (1)-(4) are all symmetric in a and c hence it suffices to deal only with $a > 4c$. Using (1) we have $4b + 4c > 20c$ and so $b > 4c$. Adding (1) and (2) and simplifying yields $8c > a + b > 4c + 4c$, a contradiction.

RESULTS

Number of Candidates: 98

Gold

WILCOX Stewart*	North Sydney Boys High School	NSW	12
SHERIDAN Nicholas	Scotch College	VIC	12
ZHAO Yiying	Penleigh and Essendon Grammar School	VIC	12
CHAN David	Sydney Grammar School	NSW	12
WHITE Gareth	Hurlstone Agricultural High School	NSW	12
KWOK Andrew	University High School	VIC	12
CHEN Zhihong	Melbourne High School	VIC	11
KELAREVA Elena	Elizabeth College	TAS	12
RAMCHEN Bobbi	Melbourne Girls Grammar School	VIC	11
WONG Alex	James Ruse Agricultural High School	NSW	12

Silver

SEYMOUR Michael	Hurlstone Agricultural High School	NSW	12
CHAN Etienne	Parramatta High School	NSW	11
CHAN Marlon	Sydney Grammar School	NSW	12
NADASI Daniel	Cranbrook School	NSW	10
SUEN Peter	North Sydney Boys High School	NSW	12
ZHAO James	University High School	VIC	12
RAO Gautam	Mentone Grammar School	VIC	12
REN Jack	Brisbane State High School	QLD	12
WANG David	Marcellin College	NSW	10
ATKINS Ross	Pembroke School	SA	11
QIN Charles	Sydney Boys High School	NSW	11
YU Sunny	Camberwell Grammar School	VIC	12
LAI Jensen	Billanook College	VIC	11
MA Marshall	James Ruse Agricultural High School	NSW	11
PASIKA Marina	St Leonard's College	VIC	11
WIJEMUNIGE Nilmini	Somerville House	QLD	11

Australian Team to participate in 2002 IMO at Glasgow, UK

Stewart Wilcox, Nicholas Sheridan, Zhao Yiying, David Chan, Gareth White, Andrew Kwok.

2003

Problems

PAPER 1 Tuesday 11 February

1. Determine all triples (p, q, r) of positive integers that satisfy:

(i) $p(q - r) = q + r$,

(ii) p, q and r are prime numbers.

2. Determine all functions f that are defined for all real numbers $x \neq 0, 1$, with real numbers as their values, and which satisfy the equation

$$f(x) + \frac{1}{2x} f\left(\frac{1}{1-x}\right) = 1.$$

3. Let ABC be a triangle such that $\angle ACB = 2\angle ABC$, and let D be a point in the interior of ABC satisfying $AD = AC$ and $DB = DC$.

Prove that $\angle BAC = 3\angle BAD$.

4. Let

$$p(x) = x^{2003} + a_{2002}x^{2002} + a_{2001}x^{2001} + \cdots + a_2x^2 + a_1x + a_0,$$

where $a_0, a_1, \cdots, a_{2002}$ are integers. Let $q(x) = (p(x))^2 - 25$.

Prove that there are not more than 2003 distinct integers m such that $q(m) = 0$.

PAPER 2 Wednesday 12 February

5. After several kilometres of a televised bicycle race along a straight stretch of road on the Nullarbor, the favourite Andrew pulled well ahead of the rest of the field closely followed by Brenda and then Chris. For the remainder of the race those three were ahead of the rest and, although they frequently changed places, at no time were all three abreast. During the finish a thunderstorm caused the TV signal to drop out, and when it came back on the race was over. The frustrated viewers only heard that the leading position changed 19 times while the third position changed 17 times and that Brenda came third.

Who won the race and why?

6. Let AD be a median of triangle ABC. Let point E lie on AD (extended if necessary) such that CE is perpendicular to AD. Suppose that angle ACE equals angle ABC.

Prove that either $AB = AC$ or angle BAC is a right angle.

7. Let $a_1, a_2, a_3, \ldots$ be a sequence defined by

(i) $a_1 = 0$,

(ii) either $a_{i+1} = a_i + 1$
or $a_{i+1} = -a_i - 1$,
for each $i \geq 0$.

An example is $0, 1, 2, 3, -4, -3, 2, \ldots$.

Prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq -\dfrac{1}{2}$, for all positive integers n.

8. Let S be any sequence of n letters ($n \geq 1$) not more than 10 of which are different, e.g. MATHEMATICIANS or GOOLLLDDMMMMEDALLLLLSYESYESYES.

Prove that each letter of this sequence can be replaced by a single decimal digit such that

(i) different letters are replaced by different digits,

(ii) the first letter of the sequence is replaced by a digit other than 0,

(iii) the resulting n-digit number is a multiple of 9.

Solution 1

Alternative 1 (Ross Atkins, Pembroke School, SA; Jason Chung, James Ruse Agricultural High School, NSW)

Solving the equation for p, then rearranging yields

$$\begin{aligned} p &= \tfrac{q+r}{q-r} \\ &= 1+\tfrac{2r}{q-r} \\ \Rightarrow \quad p-1 &= \tfrac{2r}{q-r} \\ \Rightarrow \quad (p-1)(q-r) &= 2r. \end{aligned}$$

Since r is prime, then the factors of $2r$ are $1, 2, r, 2r$. Thus $p-1$ must be one of those values, leaving us with four cases to check.

Case 1: $p-1=1$. This quickly leads to $q=3r$, which is impossible for primes q, r.

Case 2: $p-1=2$. This quickly leads to $q=2r$, which is impossible for primes q, r.

Case 3: $p-1=r$. This quickly leads to $q=r+2$. Thus $(r,p,q) = (r, r+1, r+2)$. However it is impossible for three consecutive integers to be prime.

Case 4: $p-1=2r$. This quickly leads to $q=r+1$. Thus q, r are consecutive integers which are also prime. This is possible only if $r=2$ and $q=3$, and thus $p=5$.

Clearly the triple $(p,q,r)=(5,3,2)$ satisfies the conditions of the problem.

Alternative 2 (Vinoth Nandakumar, Sydney Boys High School, NSW)

As in the first alternative we establish

$$(p-1)(q-r)=2r.$$

If $p=2$ then we quickly get $q=3r$ which is impossible. If $p \neq 2$ then p is odd and we may write $p=2k+1$ for some positive integer k. The equation now becomes

$$k(q-r)=r.$$

Thus $k|r$. Since r is prime we have $k=1$ or $k=r$. If $k=1$, then $q=2r$ which is impossible. If $k=r$, then $q=r+1$, which is possible only if $r=2$ and $q=3$ in which case $p=5$. The triple $(5,3,2)$ satisfies the criteria.

Alternative 3 (Kamil Khan, St Michael's Grammar School, VIC; Daniel Nadasi, Cranbrook School, NSW; Ivan Guo, Sydney Boys High School, NSW)

Two cases may be considered, namely where $p = 2$ and where $p \neq 2$.

Case 1: $p = 2$. This quickly yields $q = 3r$, which is impossible for primes q, r.

Case 2: $p \neq 2$. Thus p, being prime, is odd. Furthermore $\frac{p+1}{2}$ is an integer. Adding $q - r$ to both sides of the given relation yields

$$\begin{array}{rrcl} & (p+1)(q-r) & = & 2q \\ \Rightarrow & \frac{p+1}{2}(q-r) & = & q. \end{array}$$

Thus $q - r$ is a factor of q. Since q is prime and $q - r < q$, it must be that $q - r = 1$. Thus q, r are consecutive integers which are also prime. This is possible only if $r = 2$ and $q = 3$, which in turn implies $p = 5$. That this triple is valid is easily verified.

Solution 2 (Laurence Field, Sydney Grammar School, NSW; Marshall Ma, James Ruse Agricultural High School, NSW)

We notice that the function $g(x) = \frac{1}{1-x}$ forms a 3-cycle. Indeed under the application of g three times we see $x \mapsto \frac{1}{1-x} \mapsto \frac{x-1}{x} \mapsto x$ for $x \neq 0, 1$. Setting $x = y$, then $x = \frac{1}{1-y}$ then $x = \frac{y-1}{y}$ successively into the functional equation yields:

$$f(x) + \frac{1}{2x} f\left(\frac{1}{1-x}\right) = 1, \tag{1}$$

$$f\left(\frac{1}{1-x}\right) + \frac{1-x}{2} f\left(\frac{x-1}{x}\right) = 1, \tag{2}$$

and

$$f\left(\frac{x-1}{x}\right) + \frac{x}{2(x-1)} f(x) = 1. \tag{3}$$

These are three simultaneous equations in $f(x)$, $f(\frac{1}{1-x})$ and $f(\frac{x-1}{x})$. We note that $2x \times (1)$ gives

$$2xf(x) + f\left(\frac{1}{1-x}\right) = 2x. \tag{4}$$

Furthermore, $(4) - (2)$ gives

$$2xf(x) + \frac{x-1}{2} f\left(\frac{x-1}{x}\right) = 2x - 1 \tag{5}$$

and $\frac{x-1}{2} \times (3)$ gives

$$\frac{x-1}{2} f\left(\frac{x-1}{x}\right) + \frac{x}{4} f(x) = \frac{x-1}{2}. \qquad (6)$$

Then (5) – (6) gives

$$\frac{7x}{4} f(x) = 2x - 1 - \frac{x}{2} + \frac{1}{2} = \frac{3x-1}{2}.$$

Tidying this up yields

$$f(x) = \frac{6x-2}{7x}.$$

We still must check whether or not this satisfies the original equation. Indeed

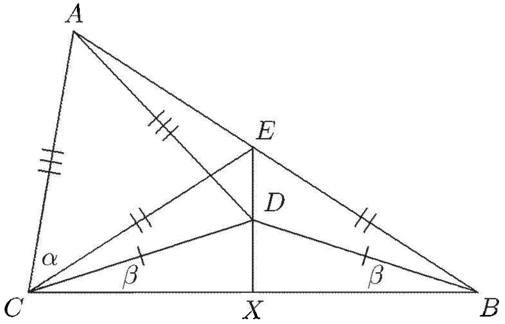
$$\begin{aligned}
\text{LHS} &= f(x) + \frac{1}{2x} f\left(\frac{1}{1-x}\right) \\
&= \frac{6x-2}{7x} + \frac{1}{2x} \cdot \frac{\frac{6}{1-x} - 2}{\frac{7}{1-x}} \\
&= \frac{6x-2}{7x} + \frac{1}{2x} \cdot \frac{2x+4}{7} \\
&= 1 \\
&= \text{RHS},
\end{aligned}$$

as required.

Solution 3

Alternative 1 (Zhihong Chen, Melbourne High School, VIC)

Let E be on AB such that CE is the angle bisector of $\angle ACB$. Thus $\angle ACE = \angle ECB = \angle ABC = \alpha$. This means that triangle CEB is isosceles. Let X be the midpoint of BC. Note that since triangles CDB and CEB are isosceles, that points D and E both lie on the perpendicular bisector of BC. Thus points X, D and E are collinear.



Now $\triangle ACE \sim \triangle ABC$ (AA). Thus

$$\begin{aligned} \frac{AC}{AE} &= \frac{AB}{AC} \\ \Rightarrow \quad \frac{AD}{AE} &= \frac{AB}{AD} \quad \text{(since } AC = AD\text{).} \end{aligned}$$

Hence $\triangle ADE \sim \triangle ABD$. Thus $\angle ADE = \angle ABD = \alpha - \beta$. With this information we can now "angle-chase" out the result. Indeed

$$\begin{aligned} \angle EDB &= \angle DXB + \angle DBX \\ &= 90^\circ + \beta \end{aligned}$$

so

$$\begin{aligned} \angle ADB &= \angle ADE + \angle EDB \\ &= \alpha - \beta + 90^\circ + \beta \\ &= \alpha + 90^\circ \end{aligned}$$

therefore

$$\begin{aligned} \angle BAD &= 180^\circ - \angle ABD - \angle ADB \\ &= 90^\circ - 2\alpha + \beta. \end{aligned}$$

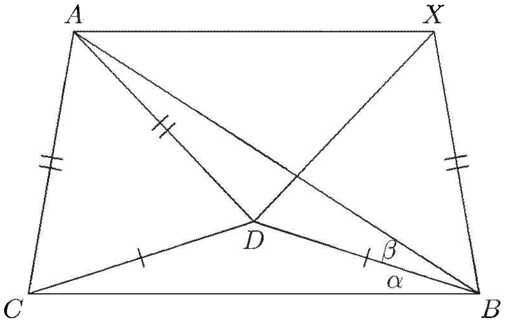
Since $\angle ADC = \angle ACD = \angle ACB - \angle DCB = 2\alpha - \beta$, we have

$$\begin{aligned} \angle CAD &= 180^\circ - 2\angle ACD \\ &= 180^\circ - 2(2\alpha - \beta) \\ &= 2(90^\circ - 2\alpha + \beta) \\ &= 2\angle BAD. \end{aligned}$$

Thus $\angle BAC = \angle CAD + \angle BAD = 3\angle BAD$ as required.

Alternative 2 (Kim Ramchen, Melbourne Grammar School, VIC)

Construct the point X so that $BX = AC$ and BA is the angle bisector of $\angle XBC$, so that $\angle XBA = \angle ABC$. Construct also the lines AX and DX.



It is easy to see that $\angle XBD = 2\angle ABC - \angle DBC = \angle ACB - \angle DCB = \angle ACD$. Thus $\triangle ACD \equiv \triangle XBD$ (SAS). Hence $DX = AX$ and the figure is symmetric about the perpendicular bisector of BC. In particular $AX \parallel CB$. Thus $\angle XAB = \angle ABC = \angle XBA$. Thus $AX = XB = AC = AD = XD$ and so triangle AXD is equilateral.

The remainder is an angle-chase. Setting $\angle ABD = \beta$ and $\angle DBC = \alpha$, we have $\angle ACB = 2(\alpha + \beta)$. Thus

$$\begin{aligned} \angle ADC &= \angle ACD \\ &= 2(\alpha + \beta) - \alpha \\ &= 2\beta + \alpha. \end{aligned}$$

Hence $\angle CAD = 180^\circ - 4\beta - 2\alpha$. We also have

$$\begin{aligned} \angle BAC &= 180^\circ - \angle ACB - \angle ABC \\ &= 180^\circ - 3\alpha - 3\beta. \end{aligned}$$

Thus

$$\begin{aligned} \angle BAD &= \angle BAC - \angle DAC \\ &= \beta - \alpha. \end{aligned}$$

Now $3\angle BAD = \angle BAC$ if and only if $3\beta - 3\alpha = 180^\circ - 3\alpha - 3\beta$ which is true if and only if $\beta = 30^\circ$. Thus it suffices to prove that $\beta = 30^\circ$. This follows quickly since from equilateral triangle ADX we have

$$\begin{aligned} 60^\circ &= \angle BAD + \angle BAX \\ &= (\beta - \alpha) + (\beta + \alpha) \qquad \text{(since } AX \parallel CB\text{)} \\ &= 2\beta \end{aligned}$$

and we are done.

Solution 4

Alternative 1 (Ross Atkins, Pembroke School, SA)

Assume that $q(x) = 0$ has at least 2004 distinct integer roots. Since $q(x) = (p(x) - 5)(p(x) + 5)$ this implies that at least one of $p(x) - 5 = 0$ or $p(x) + 5 = 0$ has at least 1002 distinct integer roots.

Suppose that $p(x) - 5 = 0$ has at least 1002 distinct integer roots. (The case for $p(x) + 5 = 0$ having at least 1002 distinct integer roots is handled similarly.) If $r_1, r_2, \ldots, r_{1002}$ are integer roots of $p(x) - 5 = 0$ then we may write

$$p(x) - 5 = s(x) \prod_{i=1}^{1002} (x - r_i).$$

If m is not a root of $p(x) - 5 = 0$ then $|s(m)| \neq 0$ so that $|s(m)| \geq 1$ and $\prod_{i=1}^{1002} (m - r_i)$, being a product of 1002 distinct integers must have magnitude at least $1^2 \cdot 2^2 \cdot \ldots \cdot 501^2 > 1000$. Thus $|p(m) - 5| > 1000$. Thus $|p(m)| > 995$ and so $p(m) + 5 = 0$ is impossible.

Thus all the roots of $q(x)$ are roots of $p(x)-5=0$. Yet we assumed that $q(x)=0$ had at least 2004 distinct integer roots, so this must also be the case for $p(x)-5=0$. This is impossible since $p(x)-5$ is a polynomial of degree 2003.

Alternative 2 (Kim Ramchen, Melbourne Grammar School, VIC; Ivan Guo, Sydney Boys High School, NSW)

Suppose that $q(x)=(p(x))^2-25=(p(x)-5)(p(x)+5)$ has more than 2003 distinct integer roots. Then since both $p(x)-5$ and $p(x)+5$ are polynomials of degree 2003, it must be the case that there is at least one integer root of $p(x)-5=0$ and at least one of $p(x)+5=0$. Let the integer roots of $p(x)-5=0$ be $s_1, s_2, \ldots, s_k$, and the integer roots of $p(x)+5=0$ be $t_1, t_2, \ldots, t_l$, where $k, l \geq 1$. Note that $k+l>2003$.

It is true for any polynomial $f(x)$ that $(x-y)$ is a factor of $f(x)-f(y)$. (This follows simply from $(x-y)|(x^n-y^n)$.) Thus s_1-t_i is a factor of $p(s_1)-p(t_i)$ for $i=1,2,\ldots,l$. Yet $p(s_1)=5$ and $p(t_i)=-5$, thus the l different numbers s_1-t_i $(i=1,2,\ldots,l)$ are all distinct factors of $5-(-5)=10$. However 10 has only 8 factors namely $\pm 1, \pm 2, \pm 5, \pm 10$. Thus $l \leq 8$. In a similar way we can prove that $k \leq 8$. Thus $k+l \leq 16$. This contradicts the earlier find that $k+l>2003$.

Comment

While both these solutions achieve the desired result, it is possible to fine tune the approach to show that if $p(x)-5=0$ and $p(x)+5=0$ both have integer roots, then $q(x)=0$ has at most 8 distinct integer roots. Thus the number 2003 in the question can in fact be replaced by a number as low as 8.

Solution 5

Alternative 1 (Sam Chow, Scotch College, VIC; Ross Atkins, Pembroke School, SA; Ting Chen, University High School, VIC; Oliver Sisson, Melbourne Grammar School, VIC)

We write A for Andrew, B for Brenda and C for Chris. Every time B switches place twice, she ends up back in second place. However B finished third. Thus B switched places an odd number of times. The switches not involving B are exactly the switches only between A and C. Since the total number of switches (36) is even, and B switched places an odd number of times, we conclude that A and C must have switched places an odd number of times. As A was ahead of C at the start, A will still be ahead of C every time they switch twice. Thus A is ahead of C after an even number of switches (between A and C) and also A is behind C after an odd number of switches (between A and C). Thus at the conclusion of the race A could not have won because he finished behind C. Furthermore we know that B did not win. Thus C finished first. Chris won the race.

Alternative 2 (Laurence Field, Sydney Grammar School, NSW; Patrick Coleman, St Peter's College, SA; Minh-Son To, St Peter's College, SA; Jason Chung, James Ruse Agricultural High School, NSW)

For convenience we represent the position at any point in the race by an ordering of the three letters A, B, C. For example the notation ABC means that Andrew is in first position, Brenda in second and Chris in third. Note that there are six different possible orderings of the three letters. We partition them into two types. We designate the orderings ABC, CAB and BCA as *right-wing* and the orderings ACB, CBA and BAC as *left-wing.*

We note that any single position change transforms a left-wing ordering into a right-wing ordering and vice versa. It immediately follows that since 36 is even and we started with a right-wing ordering, namely ABC, then we must finish with a right wing ordering. We are kindly told that B came third. The only right-wing ordering with B in third position is CAB, which tells us that Chris must have won the race.

Comment

The concepts of "right-wing" and "left-wing" are more commonly called "even" and "odd" when used in the context of describing Permutation Groups.

Alternative 3 (Marshall Ma, James Ruse Agricultural High School, NSW)

Call the starting position A,B,C. Define the move x to be the exchange of first and second positions and define the move y to be the exchange of second and third positions. Note that these are the only moves allowed in the race. Note that an odd number of consecutive x moves is the same as performing a single x move and that an even number of consecutive x moves is the same as performing no x moves. The same holds for y moves. We code a sequence of position exchanges as a sequence of letters reading from the left.

For example a sequence of moves $x, x, y, x, x, x, y, y, x, y, x, y, y, y$ can be simplified successively to y, x, x, y, x, y then y, y, x, y then finally to x, y. Since we can always delete a pair of consecutive xs or consecutive ys from any sequence, we may assume that any sequence we are considering is of the form $x, y, x, y, \ldots$ or $y, x, y, x, \ldots$. Note that any such deletions remove an even number of letters. Since there were 36 moves in the beginning, there must be an even number of letters in the sequence after simplifying deletions have been made.

We are told that B came last so that the final position is 1,2,B.

Case 1: The sequence is $x, y, x, y, \ldots, x, y$. We may work backwards, unpeeling the sequence from the right. We thus get the following sequence which repeats as a 6-cycle:

$$\begin{array}{ccccc}
1 & 2 & \mathrm{B} & \qquad & (0) \\
1 & \mathrm{B} & 2 & & (1) \\
\mathrm{B} & 1 & 2 & & (2) \\
\mathrm{B} & 2 & 1 & & (3) \\
2 & \mathrm{B} & 1 & & (4) \\
2 & 1 & \mathrm{B} & & (5) \\
 & \vdots & & &
\end{array}$$

Since B started in second position we see that only after $6k+1$ or $6k+4$ moves have been peeled off is this the case. Remembering that there have to be an even number of moves in the simplified sequence, leaves only the choice that $6k+4$ moves have been made. Thus the only valid starting position which finishes in 1,2,B is 2,B,1.

Case 2: The sequence is $y, x, y, x, \ldots, y, x$. We may work backwards, unpealing the sequence from the right. We thus get the following sequence which repeats as a 6-cycle:

$$\begin{array}{ccccc}
1 & 2 & \mathrm{B} & \qquad & (0) \\
2 & 1 & \mathrm{B} & & (1) \\
2 & \mathrm{B} & 1 & & (2) \\
\mathrm{B} & 2 & 1 & & (3) \\
\mathrm{B} & 1 & 2 & & (4) \\
1 & \mathrm{B} & 2 & & (5) \\
 & \vdots & & &
\end{array}$$

Since B started in second position we see that only after $6k+2$ or $6k+5$ moves have been peeled off is this the case. Remembering that there have to be an even number of moves in the simplified sequence, leaves only the choice that $6k+2$ moves have been made. Thus the only valid starting position which finishes in 1,2,B is 2,B,1.

We see in both cases that if the race ends as 1,2,B, then it must start as 2,B,1. Thus A=2 and C=1, which means that C wins the race.

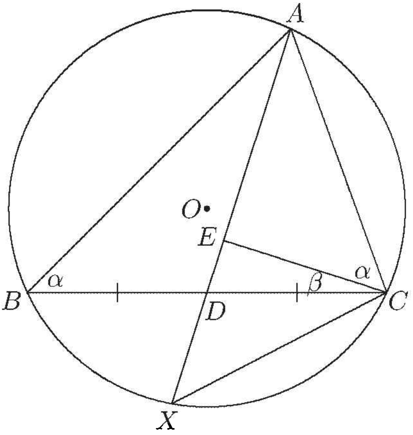
Comment

The concepts used in this proof are also to be found in an area of mathematics called *Group Theory.* One method of studying Group Theory includes the idea of writing, using only symbols from a given alphabet, sequences of letters from left to right which are called *words*, and *relations* which are also words that we are allowed to add to or delete from any place within a word. We may also *compose* words simply by juxtaposing them. In the proof above the alphabet is $\{x, y\}$. The relations we may add or delete from any word are $\{xx, yy, xyxyxy\}$. The third relation comes from the 6-cycle. Note too that the relation $yxyxyx$ is redundant and can be derived from the three relations already given. Using these relations one can show that any word is equivalent to one of the following six words $\cdot, x, xy, xyx, xyxy, xyxyx$ where "$\cdot$" represents the empty word. In general it is a very hard problem to understand the structure of a group presented in this word-relation way.

Solution 6

Alternative 1 (Sam Chow, Scotch College, VIC)

Construct the circumcircle of triangle ABC and let O be the circumcentre. Extend AD to meet the circumcircle at X. Let $\angle ABC = \angle ACE = \alpha$ and let $\angle ECD = \beta$.



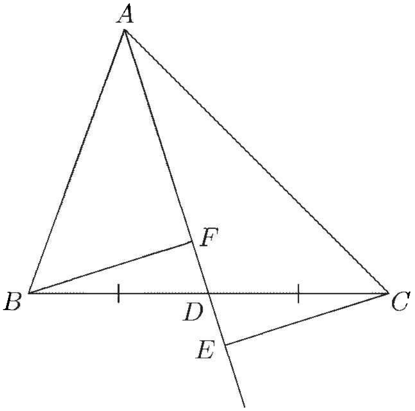
We quickly compute that $\angle ADC = 90^\circ - \beta$ and from $\angle ABD + \angle BAD = \angle ADC$ that $\angle BAD = 90^\circ - \alpha - \beta$. Thus also $\angle BCX = \angle BAX = 90^\circ - \alpha - \beta$. Hence $\angle ACX = 90^\circ - \alpha - \beta + \alpha + \beta = 90^\circ$. We conclude that AX is a diameter and thus O lies on AX. However AX also passes through D. Thus $O = D$ or $AX \perp BC$.

Case 1: If $O = D$, then BC is a diameter and $\angle BAC = 90^\circ$.

Case 2: If $AX \perp BC$ then in triangle ABC we have that the median from A coincides with the altitude from A, and thus $AB = AC$.

Alternative 2 (Laurence Field, Sydney Grammar School, NSW)

Let F be the point on AD such that $BF \perp AD$.



We use the standard notation, letting $b = AC$, $c = AB$, $a = BC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$. We are given $\angle ACE = \angle ABC = \beta$. It is easy to see that $\triangle BFD \equiv \triangle CED$ (AAS). Thus $\angle ABF = \angle ABD - \angle FBD = \beta - \angle DCE = \beta - (\beta - \gamma) = \gamma$. In triangle ABF we have $BF = c\cos\gamma$ and in triangle ACE we have $CE = b\cos\beta$. Thus

$$b\cos\beta = c\cos\gamma.$$

We simplify this using the cosine rule:

$$\cos\beta = \frac{c^2 + a^2 - b^2}{2ac}$$

and

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab}.$$

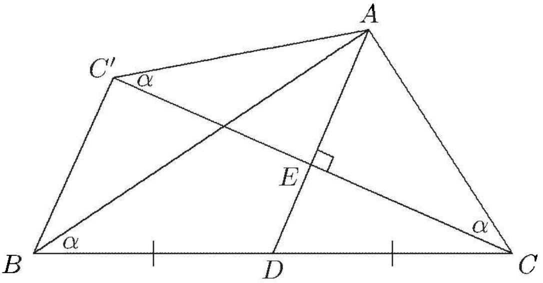
Substituting these in yields:

$$\begin{aligned} b\left(\frac{c^2 + a^2 - b^2}{2ca}\right) &= c\left(\frac{a^2 + b^2 - c^2}{2ab}\right) \\ \Rightarrow \quad b^2(c^2 + a^2 - b^2) &= c^2(a^2 + b^2 - c^2) \\ \Rightarrow \quad c^4 - a^2c^2 + a^2b^2 - b^4 &= 0 \\ \Rightarrow \quad (c^2 - b^2)(c^2 + b^2 - a^2) &= 0. \end{aligned}$$

Thus one of the two brackets is zero. If the first is zero then $AB = AC$. If the second is zero then $a^2 = b^2 + c^2$ which means that the triangle is right-angled at A by Pythagoras' Theorem.

Alternative 3 (David Wang, James Ruse Agricultural High School, NSW; Michael Chou, Brisbane Grammar School, QLD)

Set $\angle ABC = \angle ACE = \alpha$.



There are two cases to consider.

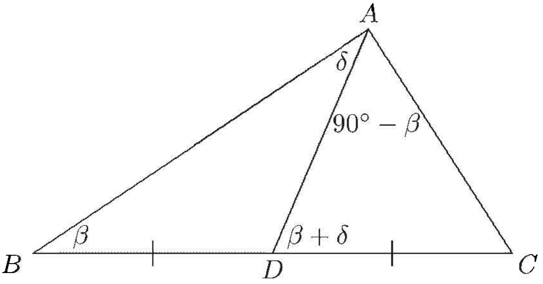
Case 1: $E = D$. This implies in triangle ABC that the altitude from A coincides with the median from A. Thus $AB = AC$.

Case 2: $E \neq D$. Reflect C about AD to C'. We note that $\angle AC'C = \angle ACE = \angle ABC$. Thus $AC'BC$ is a cyclic quadrilateral. Furthermore since E is the midpoint of CC' and D is the midpoint of BC we have that $DE \parallel BC'$. Hence $\angle BC'C = \angle DEC = 90°$. Finally since $AC'BC$ is cyclic we conclude that $\angle BAC = \angle BC'C = 90°$.

Alternative 4 (Zhihong Chen, Melbourne High School, VIC)

Using standard notation set $a = BC$, $b = AC$, $c = AB$, $\beta = \angle ABC$. Also set $\delta = \angle BAD$.

Note that $\angle ADC = \beta + \delta$. From triangle CAE we compute that $\angle DAC = \angle CAE = 180° - (90° + \beta) = 90° - \beta$. This information completely defines the problem and is sufficient for our purposes. In fact point E has become redundant.



Applying the sine rule to triangle ABC yields

$$\frac{b}{\sin\beta} = \frac{a}{\sin(90° - \beta + \delta)}$$

$$\Rightarrow \quad \frac{b}{a} = \frac{\sin\beta}{\cos(\beta-\delta)}.$$

Applying the sine rule to triangle ADC yields

$$\begin{aligned} \frac{b}{\sin(\beta+\delta)} &= \frac{\frac{1}{2}a}{\sin(90^\circ-\beta)} \\ \Rightarrow \quad \frac{b}{a} &= \frac{\sin(\beta+\delta)}{2\cos\beta}. \end{aligned}$$

Equating these two expressions for $\frac{b}{a}$ yields

$$\begin{aligned} 2\sin\beta\cos\beta &= \sin(\beta+\delta)\cos(\beta-\delta) \\ \Rightarrow \quad \sin 2\beta &= \tfrac{1}{2}(\sin 2\beta + \sin 2\delta) \qquad \text{(products to sums)} \\ \Rightarrow \quad \sin 2\beta &= \sin 2\delta. \end{aligned}$$

Hence we have two cases: $2\beta = 2\delta$ or $2\beta = 180^\circ - 2\delta$. The first case tells us that $\beta = \delta$ so that $\angle BAC = \delta + 90^\circ - \beta = 90^\circ$. The second case tells us that $\beta + \delta = 90^\circ$ so that $\angle ADB = 90^\circ$, causing AD to be an altitude as well as a median from A, and thus $AB = AC$.

Solution 7

Alternative 1 (Ross Atkins, Pembroke School, SA)

We note that the second condition is equivalent to $a_{i+1}^2 = (a_i+1)^2 \ldots (*)$. After some experimentation we conjecture that $2\sum_{i=1}^{n} a_i = -n + (a_n + 1)^2$. This is quite straightforward to prove by induction as follows:

For $n = 1$ we know $a_1 = 0$ and LHS $= 0$ and RHS $= -1 + (0+1)^2 = 0 =$ LHS.

Assume it is true for $n = k$. Thus $2\sum_{i=1}^{k} a_i = -k + (a_k + 1)^2$. For $n = k+1$, we have

$$\begin{aligned} \text{LHS} &= 2\sum_{i=1}^{k+1} a_i \\ &= 2\sum_{i=1}^{k} a_i + 2a_{k+1} \\ &= -k + (a_k+1)^2 + 2a_{k+1} \qquad \text{(inductive assumption)} \\ &= -(k+1) + a_{k+1}^2 + 2a_{k+1} + 1 \qquad \text{(from (*))} \\ &= -(k+1) + (a_{k+1}+1)^2 \\ &= \text{RHS}. \end{aligned}$$

This completes the proof of the conjecture.

We now obtain from this that $2\sum_{i=1}^{n} a_i \geq -n$ which immediately implies that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq -\frac{1}{2},$$

as required. Note equality occurs if and only if $a_n = -1$.

Alternative 2 (Zhihong Chen, Melbourne High School, VIC)

This solution is very similar to that of *Alternative 1.*

Starting with the relation $a_{i+1}^2 = (a_i+1)^2$ we find that $2a_i+1 = a_{i+1}^2 - a_i^2$. Summing this for $i = 1$ to n we find that it telescopes to

$$\begin{array}{rcl} 2a_1 + 1 + 2a_2 + 1 + \cdots + 2a_n + 1 &=& a_{n+1}^2 - a_1^2 \\ &=& a_{n+1}^2 \\ &\geq& 0 \\ \Rightarrow \qquad \dfrac{a_1 + a_2 + \cdots + a_n}{n} &\geq& -\frac{1}{2} \end{array}$$

as required.

Alternative 3 (Laurence Field, Sydney Grammar School, NSW; Daniel Nadasi, Cranbrook School, NSW)

For ease of argument we construct a new sequence $b_1, b_2, \ldots$ defined by $b_i = 2a_i + 1$, i.e. $a_i = \frac{1}{2}(b_i - 1)$. Note that $b_1 = 2a_1 + 1 = 1$ and that

$$b_{i+1} = 2a_{i+1} + 1 = \left\{ \begin{array}{c} 2a_i + 3 \\ \text{or} \\ -2a_i - 1 \end{array} \right\} = \left\{ \begin{array}{c} b_i + 2 \\ \text{or} \\ -b_i \end{array} \right\}.$$

So the new sequence $b_1, b_2, \ldots$ satisfies

(i) $b_1 = 1$;

(ii) $b_{i+1} = b_i + 2$ or $b_{i+1} = -b_i$ for all $i > 0$.

The equivalent problem we need to prove is

$$\begin{array}{rrcl} & \dfrac{\sum_{i=1}^n a_i}{n} &\geq& -\frac{1}{2} \\ \Leftrightarrow & n + 2\sum_{i=1}^n a_i &\geq& 0 \\ \Leftrightarrow & \sum_{i=1}^n (2a_i + 1) &\geq& 0 \\ \Leftrightarrow & \sum_{i=1}^n b_i &\geq& 0. \end{array}$$

This is simpler than the original problem. We note that the passage from b_i to b_{i+1} is either an increment of 2 or a change of sign.

Now consider any time a sign reversal occurs from negative to positive, e.g. $\ldots$, $-m-2$, $-m$, m, $m+2$, $\ldots$. We can augment this sequence by inserting in between $-m$ and m all the odd integers in order from $-m+2$ to $m-2$. (If $m = 1$ we do nothing.) This augmented sequence is still a valid sequence, and still has the same sum. So by applying this augmentation procedure we can ignore all sequences with negative to positive sign changes.

From what we have constructed, every sequence now considered must have the form 1, 3, …, n_1, $-n_1$, $-n_1+2$, …, n_2, $-n_2$, $-n_2+2$, …, where $n_i > 0$ for $i = 1, 2, \ldots$. For the purposes of summation, the sequence will stop at some number n.

We now divide the sequence into segments as small as possible such that each segment starts with 1 and ends with -1. (The final segment may or may not end with -1, but the others must.) We find that all the segments except perhaps the last are of the form $1, 3, 5, \ldots k, -k, -k+2, \ldots, -3, -1$. In each of these segments every positive term has a corresponding negative term. So the sum of the numbers in each of these segments is zero. Hence the some of the whole sequence equals the sum of the last fragment. The last fragment can be thought of like any other segment except with perhaps a few negative terms missing off the end. Thus the sum of the numbers in the last fragment is at least zero and equals zero if and only if the last member of the sequence is -1. Thus we have proven that $b_1 + b_2 + \cdots + b_n \geq 0$ with equality if and only if $b_n = -1$. This establishes the original inequality. Equality occurs if and only if $a_n = -1$.

Alternative 4 (Problems Committee; Jacky Poon, Baulkham Hills High School, NSW)

Note that the expression $\frac{a_1+a_2+\cdots+a_n}{n}$ is really the average of the first n terms. We call a finite sequence $(a_1, a_2, \ldots, a_n)$, *offending* if it satisfies the criteria but has average less than $-\frac{1}{2}$.

Suppose now an offending sequence exists. Then there exists an offending sequence $S = (a_1, a_2, \ldots, a_m)$ of shortest length. We now show how to construct a smaller offending sequence, thus obtaining a contradiction and solving the problem.

Since the average of S is negative, then S must contain a negative term. Let a_k be the first such negative term. Then $a_{k-1} = -a_k - 1 \geq 0$. Note that the average of a_{k-1} and a_k equals $-\frac{1}{2}$. This shows that if we remove the terms a_{k-1} and a_k from S we obtain a shorter sequence S' with an average still less than $-\frac{1}{2}$. That S' also satisfies the criteria must be checked, however this is routine. (There are four cases.) Thus S' is an offending sequence of shorter length and we have obtained the required contradiction.

Solution 8 (Problems Committee)

Condition (iii) means that the digits of the number to be obtained add up to a multiple of 9. Note that condition (ii) can be ignored because 0 and 9 can be interchanged without altering the divisibility by 9 of the number already obtained. We may assume that the given sequence has exactly 10 different letters some of which possibly occur zero times.

The crucial idea we use is to assign 9 to some fixed letter, say which occurs q times, and assign the numbers 0 to 8 arbitrarily to the remaining letters. We end up with a number w. Consider the transformation s on w which we describe as replacing in w: 0 by 1, 1 by 2, …, 7 by 8 and 8 by 0 to form $s(w)$. Let $d = s(w) - w$. Note that $d \equiv n - q \pmod 9$. We have $s(w) \equiv w + d \pmod 9$. Also $s^{(2)}(w) = s(s(w)) \equiv s(w) + d \equiv w + 2d \pmod 9$. Continuing this process we have in general $s^{(i)}(w) \equiv w + id \pmod 9$. We want to prove that for some i we have

$$w + id \equiv 0 \pmod 9. \qquad (*)$$

The congruence (*) is certainly solvable if d is invertible (mod 3), i.e. if $d \not\equiv 0 \pmod 3$. Remembering that $d \equiv n - q \pmod 9$, if some assignment of the number 9 to some fixed letter results in $n - q \not\equiv 0 \pmod 3$ then we are done.

Suppose now that all assignments of 9 to some fixed letter result in $n - q \equiv 0 \pmod 3$. This means that every letter in S occurs exactly a number of times that is congruent to n (mod 3). Thus in whatever valid way we assign digits to the letters of S, the resulting number w will be be such that $w \equiv n(0 + 1 + 2 + \cdots + 9) \equiv 0 \pmod 3$. Going back to the congruence (*) we may now divide through by 3 to obtain the equivalent equation

$$\frac{w}{3} + i\left(\frac{d}{3}\right) \equiv 0 \pmod 3. \qquad (**)$$

The congruence (**) is certainly solvable if $\frac{d}{3}$ is invertible (mod 3), i.e., if $\frac{d}{3} \not\equiv 0 \pmod 3$. Remembering that $d \equiv n - q \pmod 9$, if some assignment of the number 9 to some fixed letter results in $n - q \not\equiv 0 \pmod 9$ then we are done.

Suppose now that all assignments of 9 to some fixed letter result in $n - q \equiv 0 \pmod 9$. This means that every letter in S occurs exactly a number of times that is congruent to n (mod 9). Thus in whatever valid way we assign digits to the letters of S, the resulting number w will be be such that $w \equiv n(0 + 1 + 2 + \cdots + 9) \equiv 0 \pmod 9$. In this case every valid assignment results in a number which is divisible by 9.

RESULTS

Number of Candidates: 104

Gold

CHEN Zhihong	Melbourne High School	VIC	12
GUO Ivan	Sydney Boys High School	NSW	11
NADASI Daniel	Cranbrook School	NSW	11
ATKINS Ross	Pembroke School	SA	12
FIELD Laurence	Sydney Grammar School	NSW	11
MA Marshall	James Ruse Agricultural High School	NSW	12
CHUNG Jason	James Ruse Agricultural High School	NSW	12
RAMCHEN Kim	Melbourne Grammar School	VIC	10
COLEMAN Patrick	St Peters College	SA	11
CHOW Sam	Scotch College	VIC	10
WANG David	James Ruse Agricultural High School	NSW	11

Silver

LAI Jensen	Billanook College	VIC	12
STOROZHEV Michael	Radford College	ACT	10
CHEN Ting	University High School	VIC	12
CHAN Etienne	Parramatta High School	NSW	12
WANG Jennifer	MLC School Burwood	NSW	12
KHAN Kamil	St Michael's Grammar School	VIC	11
POON Jacky	Baulkham Hills High School	NSW	12
QIN Charles	Sydney Boys High School	NSW	12
NANDAKUMAR Vinoth	Sydney Boys High School	NSW	8
LI Charles	Sydney Grammar School	NSW	9
LEONG Chris	James Ruse Agricultural High School	NSW	10
DAI Pei	James Ruse Agricultural High School	NSW	11

CHOU Michael	Brisbane Grammar School	QLD	12
MACLAURIN Dougal	Hale School	WA	12
JOESTER Jenna	Roseville College	NSW	11

Australian Team to participate in 2003 IMO at Tokyo, Japan

Zhihong Chen, Ivan Guo, Daniel Nadasi, Ross Atkins, Laurence Field, Marshall Ma.

2004

Problems

PAPER 1 Tuesday 10 February

1. Determine all pairs (a, b) of real numbers for which the equation $x^3 + 3x^2 + ax + b = 0$ has three different real solutions that can be arranged in arithmetic progression (that is, the third minus the second is equal to the second minus the first).

2. Suppose $0 \leq x \leq a \leq y \leq b \leq z$ and $a + b + x + y + z = 2004$.

Determine, with proof, the minimum possible value of $x + y + z$. Determine, with proof, the maximum possible value of $x + y + z$.

3. Determine the number of sequences $a_1, a_2, \ldots, a_{2004}$ which are the numbers $1, 2, \ldots, 2004$ in some order and satisfy

$$|a_1 - 1| = |a_2 - 2| = \ldots = |a_{2004} - 2004| > 0.$$

4. Let ABC be an equilateral triangle, and let D be a point on AB between A and B. Next, let E be a point on AC with DE parallel to BC. Further, let F be the midpoint of CD and G the circumcentre of triangle ADE.

Determine the angles of triangle BFG.

PAPER 2 Wednesday 11 February

5. Determine all non-negative integers m and n for which $6^m + 2^n + 2$ is a perfect square.

6. Decide whether or not there is a function f defined for all positive integers and taking positive integers as values such that

$$\begin{aligned} f(f(1)) &= 5, \\ f(f(2)) &= 6, \\ f(f(3)) &= 4, \\ f(f(4)) &= 3, \\ f(f(n)) &= n+2 \end{aligned}$$

for $n \geq 5$.

7. A necklace is made from an even number, $k \geq 4$, of beads, each of which is coloured red, blue or green. There is an equal number of blue beads and green beads on the necklace. It is impossible to cut the necklace into two separate strings each of which contains a positive even number of beads and each of which contains the same number of blue and green beads.

Find all the possibilities for the number of red beads on the necklace.

8. Let $ABCD$ be a parallelogram. Suppose there exists a point P in the interior of $ABCD$ such that $/ABP = 2/ADP$ and $/DCP = 2/DAP$.

Prove that $AB = BP = CP$.

Solution 1 (Vinoth Nandakumar, James Ruse Agricultural High School, NSW)

Let the three solutions be $\alpha - z$, α and $\alpha + z$ where $z > 0$. Then by the relation between the roots and the coefficients (Viete's formulae)

$$\alpha + (\alpha - z) + (\alpha + z) = -3, \tag{1}$$

$$\alpha(\alpha - z) + \alpha(\alpha + z) + (\alpha - z)(\alpha + z) = a \tag{2}$$

and

$$\alpha(\alpha - z)(\alpha + z) = -b. \tag{3}$$

From (1) we immediately see that $\alpha = -1$. Rearranging (2) and using $\alpha = -1$ yields $a = 3 - z^2$ and hence $a \leq 3$. Finally using (3) we see $b = 1 - z^2 = a - 2$. Thus if $x^3 + 3x^2 + ax + b = 0$ has three real solutions in arithmetic progression, we have shown that $b = a - 2$ and $a \leq 3$. Conversely, if $b = a - 2$ and $a \leq 3$, take $z = +\surd 3 - a$ and we easily check that $-1 - z$, -1 and $-1 + z$ are the solutions to the cubic. Thus the required answer is $(a, b) \in \{(t, t - 2) : t < 3\}$.

Solution 2

Alternative 1 (Sam Chow, Scotch College, VIC)

Since $y + z \geq a + b$ (from $y \geq a$ and $z \geq b$), we have $2(y + z) \geq a + b + y + z = 2004 - x$. Thus $y + z \geq 1002 - \frac{x}{2}$. Hence $x + y + z \geq 1002 + \frac{x}{2} \geq 1002$. This value may be achieved when $x = 0$ and $a = y = b = z = 501$.

Also $x + y + z \leq a + b + x + y + z = 2004$ (since $a, b \geq 0$). This value may be achieved when $x = a = y = b = 0$ and $z = 2004$.

Thus 1002 and 2004 are the minimal and maximal values of $x + y + z$.

Alternative 2 (Graham White, James Ruse Agricultural High School, NSW)

Since we are given that $y \geq a$, $z \geq b$ and $x \geq 0$, it follows that $a + b \leq x + y + z$. So if $x + y + z < 1002$ then it follows that $a + b < 1002$, and hence $a + b + x + y + z < 2004$. A contradiction. Thus $x + y + z \geq 1002$. Note that if $a = x = y = 0$ and $b = z = 1002$ then $x + y + z = 1002$. Thus minimum possible value for $x + y + z$ is 1002.

Now for the maximum. Since $a + b + x + y + z = 2004$ it is clear that $x + y + z \leq 2004$. Note that if $a = b = x = y = 0$ and $z = 2004$, then $x + y + z = 2004$. Thus the maximum possible value for $x + y + z$ is 2004.

Comment

Laurence Field, Sydney Grammar School, NSW, also proved that the maximum can only occur when $x = a = y = b = 0$ and $z = 2004$. He also proved that the minimum occurs precisely when $x = 0$, $a = y = t$ and $b = z = 1002 - t$ where t is any real parameter such that $0 \leq t \leq 501$.

Solution 3 (Laurence Field, Sydney Grammar School, NSW; Ivan Guo, Sydney Boys High School, NSW)

Let $S = \{1, 2, \ldots, 2004\}$. Write the sequence as a function $f : S \to S$, $f(n) = a_n$. We note some properties of valid functions f.

1. $f(i) \neq i$ for any i.
2. $f(i) \neq f(j)$ if $i \neq j$.
3. $f(i) = j$ implies $f(j) = i$.

The first property follows from $|a_i - i| \neq 0$. The second property is true because the a_i are a permutation of S. The third property can be proved as follows: Assume that $f(j) \neq i$. Suppose $j > i$ and let $k = |j - i|$. Then $f(i) = i + k$. Since $f(j) = f(i+k) = i+k \pm k$, since $f(i+k) \neq i$ by assumption then it must be that $f(i+k) = i+2k$. We now play a similar game with $f(i+2k)$, namely $f(i+2k) = i+k$ or $i+3k$. Since we already have $f(i) = i + k$, then by property 2 above it must be that $f(i + 2k) = i + 3k$. In general $f(i + nk) = i + (n - 1)k$ or $i + (n + 1)k$ and since $f(i+(n-2)k) = i+(n-1)k$, then property 2 forces $f(i+nk) = i+(n+1)k$. However then this forces f to attain ever increasing values. This is impossible since we are limited to the finitely many integers between 1 and 2004. A similar argument holds if $j < i$ regarding ever decreasing values. Thus we have established property 3.

The three properties 1, 2 and 3 together prove that f is an involution on S with no fixed points. Thus f partitions S into 1002 distinct pairs (i, j) such that $i = f(j)$ and $j = f(i)$.

Moreover since we know $|a_i - i| = k$ $(k \geq 1)$ for all $i \in S$, then given such a k there is at most one suitable function f. This is because we must have $f(1) = k + 1$, $f(2) = k + 2$, …, $f(k) = 2k$. But then since f is an involution, the values of f on $k+1$, $k+2$, …, $2k$ are already determined, namely $f(k + 1) = 1$, $f(k + 2) = 2$, …, $f(2k) = k$.

Thus f is a permutation of the block of integers from 1 to $2k$. Using similar reasoning it follows that $f(2k + 1) = 3k + 1$, …, $f(3k) = 4k$ and then $f(3k+1) = 2k+1$, …, $f(4k) = 3k$. So the next block of $2k$ integers is also closed under f.

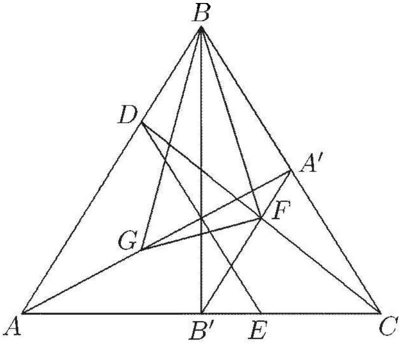
Continuing this process we see that f partitions S into blocks of $2k$ elements each of which is itself closed under the application of f. This implies that $2k|2004$ and so $k|1002$.

Furthermore for each such k we can construct the required f as described above. Thus the number of valid permutations is just the number of positive divisors of 1002. Since $1002 = 2 \cdot 3 \cdot 167$ we see that the answer is 8.

Solution 4

Alternative 1 (Patrick Coleman, St Peter's College, SA)

Let A', B' be the midpoints of BC, AC respectively. Then A, G, A' are collinear since triangle ADE is equilateral. Also note that B', F, A' are collinear since they are all halfway up the height of triangle ABC from the base AB.

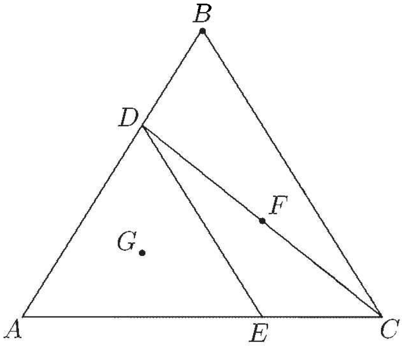


We claim that triangles BFB' and BGA are similar. Indeed $B'F = \frac{1}{2}AD$ and $AD = \sqrt{3}AG$. Note too that $BB' = \frac{\sqrt{3}}{2}AB$. Thus $\frac{BB'}{AB} = \frac{B'F}{AG} = \frac{\sqrt{3}}{2}$. Furthermore we have equality of the included angles via $\angle BB'F = 30^\circ = \angle BAG$.

Hence $\angle FBB' = \angle GBA$. Thus $\angle FBG = \angle B'BA = 30^\circ$. Finally $\angle FA'G = 30^\circ = \angle FBG$ so that $FA'BG$ is a cyclic quadrilateral. Thus $\angle GFB = \angle GA'B = 90^\circ$ and $\angle FGB = \angle FA'C = 60^\circ$. Thus we have computed the angles of triangle BFG.

Alternative 2 (Daniel Nadasi, Cranbrook School, NSW)

We may toss the figure onto the complex number plane and rescale so that $A = 0$ and $C = 1$. Then $B = \operatorname{cis}\frac{\pi}{3} = \cos 60^\circ + i\sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2}i$.



Since D lies on AB we may write $D = r\text{cis}\,\frac{\pi}{3}$ where $0 < r < 1$. Then $F = \frac{1}{2}(B + C) = \frac{1+r\text{cis}\,\frac{\pi}{3}}{2}$. Since triangle ADE is equilateral we know that $E = r$ and G being the centroid of ADE implies that $G = \frac{1}{3}(A + D + E) = \frac{1}{3}(r\text{cis}\,\frac{\pi}{3} + r)$.

We claim that triangle BGF is a 30°-60°-90° triangle. To prove this it suffices to show that BF is obtained by rotating GF through an angle of 90° clockwise around F and by dilating by a factor of $\sqrt{3}$. This is equivalent to showing that $B - F = -i\sqrt{3}(G - F)$. Indeed it is a simple matter to compute that they both equal $-\frac{r}{4} + i\frac{\sqrt{3}}{2}(1 - \frac{r}{2})$.

Solution 5

Alternative 1 (Kamil Khan, St Michael's Grammar School, VIC)

Case 1: $m = 0$.

Thus 2^n+3 is a perfect square. Note that squares (i.e. quadratic residues) can only be congruent to 0 or 1 $\pmod 4$. Thus $2^n + 3 \equiv$ 0 or 1 $\pmod 4$, i.e. $2^n \equiv$ 1 or 2 $\pmod 4$. However $2^n \equiv 0 \pmod 4$ if $n \geq 2$. Thus $n < 2$. If $n = 0$ then $2^n + 3 = 4$ which is a square. If $n = 1$ then $2^n + 3 = 5$ which is not a square. Thus this case gives $(m, n) = (0, 0)$.

Case 2: $n = 0$.

Thus $6^m + 3$ is a perfect square. Thus $6^m + 3 \equiv$ 0 or 1 $\pmod 4$, i.e. $6^m \equiv$ 1 or 2 $\pmod 4$. But $6^m \equiv 0 \pmod 4$ for $m \geq 2$. So $m < 2$. We have dealt with $m = 0$ in case 1. Finally if $m = 1$ then $6^m + 3 = 9$ which is a square. So this case gives $(m, n) = (1, 0)$.

Case 3: $m, n \geq 1$.

Note that $6^m + 2^n + 2 = 2(3^m \cdot 2^{m-1} + 2^{n-1} + 1)$. If the expression is a perfect square, then since it is even it must be divisible by 4. Thus $3^m \cdot 2^{m-1} + 2^{n-1} + 1 \equiv 0 \pmod 2$, i.e.

$$2^{m-1} + 2^{n-1} \equiv 1 \pmod 2.$$

This implies that exactly one of m or n equals 1 and the other is greater than 1.

If $n = 1$ then $6^m + 2^n + 2 \equiv (-1)^m + 2 + 2 \equiv 3$ or $5 \pmod 7$. However this is impossible since squares modulo 7 are 0, 1, 2 or 4.

If $m = 1$ then $6^m + 2^n + 2 = 2^n + 8 = 4(2^{n-2} + 2)$. Thus $2^{n-2} + 2$ must be a perfect square. If $n \geq 4$ then we note that $2^{n-2} + 2$ even but never divisible by 4 which is impossible for a square. If $n = 2$ then $2^{n-2}+2 = 3$ which is not a square. If $n = 3$ then $2^{n-2} + 2 = 4$ which is a square. So this case gives $(m, n) = (1, 3)$.

In conclusion the full solution set is $(m, n) = (0, 0)$, $(1, 0)$ or $(1, 3)$.

Alternative 2 (Laurence Field, Sydney Grammar School, NSW)

Suppose firstly that $m, n \geq 2$. Then $6^m + 2^n + 2 \equiv 2 \pmod 4$. This is impossible since squares are congruent to 0 or 1 modulo 4.

Suppose that one of m or n equals 0 and the other is at least 2. Then $6^m + 2^n + 2 \equiv 3 \pmod 4$. This is also impossible.

We may individually check the cases $(m, n) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Only the cases $(m, n) = (0, 0)$ and $(1, 0)$ yield squares.

Finally it remains only to check the cases $m = 1$, $n \geq 2$ and $n = 1$, $m \geq 2$.

In the first instance we want 2^n+8 to be a square. If $n \geq 4$ then $2^n+8 \equiv 8 \pmod{16}$. This is impossible since squares are congruent to 0, 1, 4 or 9 modulo 16. Finally we check the cases $n = 2$ and $n = 3$ individually. Only $n = 3$ yields as square. Thus $(m, n) = (1, 3)$ is a solution.

In the second instance we want $6^m + 4$ to be a square. Put $6^m + 4 = x^2$. Then $3^m \cdot 2^m = (x + 2)(x - 2)$. Now 3 divides at most one of $x + 2$ or $x - 2$. Thus the other factor which is not divisible by 3, must itself be a power of 2. Thus one factor equals $3^m \cdot 2^a$ and the other equals 2^{m-a} for some integer $0 \leq a \leq m$. Since $3^m \cdot 2^a \geq 3^m > 2^m \geq 2^{m-a}$, it must be that

$$\begin{array}{rcl} x+2 &=& 3^m \cdot 2^a \quad \text{and} \\ x-2 &=& 2^{m-a}. \end{array}$$

But $2|x - 2$ if and only if $2|x + 2$. So $a \geq 1$. Thus

$$\begin{array}{rrcl} & x-2 &=& 2^{m-a} \\ && \leq & \frac{1}{2} \cdot 2^m \\ && < & \frac{1}{2} \cdot 3^m \cdot 2^a \\ && = & \frac{1}{2}(x+2) \\ \Rightarrow & x &<& 6\,. \end{array}$$

Thus it suffices to check only $(x - 2) = 1, 2, 3$. However for each of these values of $(x - 2)$ we find that none of $(x - 2)(x + 2)$ is a power of 6. So there are no solutions in this case.

Thus the summary of solutions is $(m, n) = (0, 0)$, $(1, 0)$ and $(1, 3)$.

Solution 6 (Laurence Field, Sydney Grammar School, NSW; Reuben Fletcher-Costin, The Friends School, TAS)

We use the notation $f^{(2)}(x)$ to mean $f(f(x))$ and $f^{(4)}(x)$ to mean $f(f(f(f(x))))$. Generally $f^{(n)}(x)$ denotes $\underbrace{f(f(\ldots f(}_{n} x \underbrace{)\ldots))}_{n}$.

Note that

$$\begin{array}{rcccl} f^{(4)}(1) & = & f^{(2)}(5) & = & 7, \\ f^{(4)}(2) & = & f^{(2)}(6) & = & 8, \\ f^{(4)}(3) & = & f^{(2)}(4) & = & 3, \\ f^{(4)}(4) & = & f^{(2)}(3) & = & 4, \quad \text{and} \\ f^{(4)}(n) & = & f^{(2)}(n+2) & = & n+4 \quad (\text{for } n \geq 5). \end{array}$$

Thus the only fixed points of $f^{(4)}(x)$ occur at $x = 3$ and $x = 4$. However since $f^{(5)}(3) = f(3)$ we see that $x = f(3)$ is a fixed point of $f^{(4)}(x)$. Thus either $f(3) = 3$ or $f(3) = 4$.

If $f(3) = 3$, then $f(f(3)) = f(3) = 3 \neq 4$ which is a contradiction.

If $f(3) = 4$, then $f(4) = f(f(3)) = 4$ so that $f(f(4)) = f(4) = 4$, another contradiction.

Thus there is no such function.

Solution 7 (Laurence Field, Sydney Grammar School, NSW; Ivan Guo, Sydney Boys High School, NSW; James Wan, Scotch College, VIC)

First we note that if one string contains an equal number of blue beads as green beads and also contains an even number of beads, then so does the other. Thus it suffices to focus attention on one string only. Since the stated cut is impossible, this means that:

1. no two red beads can be adjacent, and
2. a green bead cannot be adjacent to a blue bead.

The reason is that in either case above, the two mentioned beads could be split off the necklace forming a string with the forbidden conditions.

Since any valid necklace contains both blue and green beads, pick such a pair of beads with minimal separation. Since the separation is minimal there can only be red beads in between. By the two conditions above, there is exactly one red bead between the green bead and the blue bead which we notate as GRB.

Note that the string cannot continue with a blue bead to the left or a green bead to the right by the second condition. Furthermore if the string continues with a red bead on either side then we have either RGRB or GRBR, both of which are forbidden strings (unless this is the whole string). Thus the necklace must continue GGRBB. Since the necklace is finite there must be a positive integer n such that the necklace contains $\underbrace{\mathrm{GG}\ldots\mathrm{G}}_{n}\mathrm{R}\underbrace{\mathrm{BB}\ldots\mathrm{B}}_{n}$ where there are n green beads and n blue beads either side of the red bead, but not simultaneously $n+1$ green beads and $n+1$ blue beads either side of the red bead. Without loss of generality assume that there are exactly n green beads to the left of the red bead. That means that the next bead to the left of the last green bead cannot be green, nor cannot it be blue by condition 2. Thus it must be red and the string looks like $\mathrm{R}\underbrace{\mathrm{GG...G}}_{n}\mathrm{R}\underbrace{\mathrm{BB...B}}_{n}$. However now this part of the necklace may be split off as a forbidden string except in the case that it is the whole necklace. Thus it must be the whole necklace and there are exactly two red beads.

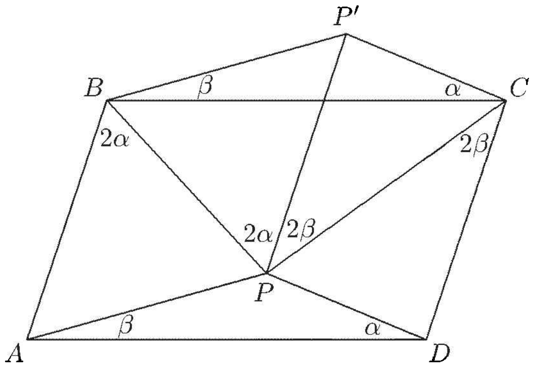
We still must verify that the necklace $\mathrm{R}\underbrace{\mathrm{GG...G}}_{n}\mathrm{R}\underbrace{\mathrm{BB...B}}_{n}$ contains no forbidden string. Suppose there is a forbidden string. Then one of the two strings contains at most one red bead. If it contains no red bead then all its beads must be the same colour and hence is not a forbidden string. If it contains one red bead then since the length of the string is even, it has an odd total of blue and green beads combined and thus it cannot contain the same amount of blue beads as green beads.

Solution 8

Alternative 1 (Daniel Nadasi, Cranbrook School, NSW)

Let $\angle DAP = \beta$ and $\angle ADP = \alpha$. Then also $\angle DCP = 2\beta$ and $\angle ABP = 2\alpha$.

Consider the translation which sends A to B. Note that it also sends D to C. Let P' be the image of P under the translation. Thus triangles APD and $BP'C$ are congruent. Hence $\angle BCP' = \alpha$ and $\angle CBP' = \beta$. Also since $APP'B$ and $DPP'C$ are parallelograms we also have that $\angle P'PB = \angle ABP = 2\alpha$ and $\angle P'PC = \angle PCD = 2\beta$.



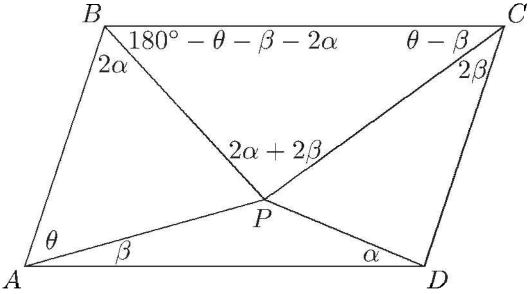
Now consider the locus of points Q such that $\angle P'QC = 2\beta$. This is a pair of circular arcs, each circular arc standing on one side of the segment $P'C$. Similarly the locus of points Q such that $\angle P'QB = 2\alpha$ is also a pair of circular arcs, each circular arc standing on one side of the segment

$P'B$. Given that we know that P is inside the parallelogram, we need only consider the pair of arcs which face back towards the parallelogram.

We know that this pair of arcs intersects at most twice. One place is at P' the other is at P. Thus P is unique. Note that if O is the circumcentre of triangle $BP'C$ then $\angle P'OC = 2\angle P'BC = 2\beta$ and $\angle P'OB = 2\angle P'CB = 2\alpha$. So O also satisfies the same criteria as P. Since P is unique we conclude that $P = O$. Thus P is the circumcentre of triangle $BP'C$. Thus $PB = PP' = PC$. We know that $PP' = AB$. Thus $AB = BP = CP$.

Alternative 2 (Kim Ramchen, Melbourne Grammar School, VIC; Ivan Guo, Sydney Boys High School, NSW)

Let $\angle DAP = \beta$ and $\angle ADP = \alpha$ as in the first solution Then again we have $\angle DCP = 2\beta$ and $\angle ABP = 2\alpha$. We also label $\angle BAP = \theta$. Since $CB \parallel DA$ we have $\angle CBA = 180^\circ - \angle DAB$. Thus $\angle CBP = 180^\circ - \theta - \beta - 2\alpha$. Since $\angle DAB = \angle DCB$ we have $\angle BCP = \theta - \beta$.



We now know enough information to begin a trigonometric approach using the sine rule in triangles APD, ABP and BCP. Let $AP = a$ and $BP = b$. The sine rule in triangle APD yields

$$\frac{AD}{\sin \angle APD} = \frac{AP}{\sin \angle ADP}.$$

This simplifies to

$$AD = \frac{a \sin(\alpha + \beta)}{\sin \alpha}. \tag{1}$$

The sine rule in triangle BCP yields $\frac{BC}{\sin \angle BPC} = \frac{BP}{\sin \angle BCP}$. This simplifies to

$$BC = \frac{b \sin(2\alpha + 2\beta)}{\sin(\theta - \beta)}. \tag{2}$$

The sine rule in triangle ABP yields $\frac{AP}{\sin \angle ABP} = \frac{BP}{\sin \angle BAP}$. This simplifies to

$$b = \frac{a \sin \theta}{\sin 2\alpha}. \tag{3}$$

From $AD = BC$ (parallelogram) using (1) and (2), we conclude that

$$\frac{a\sin(\alpha+\beta)}{\sin\alpha} = \frac{b\sin(2\alpha+2\beta)}{\sin(\theta-\beta)}. \tag{4}$$

We now substitute (3) into (4) and compute as follows:

$$\begin{aligned}
& \frac{a\sin(\alpha+\beta)}{\sin\alpha} &=& \frac{a\sin\theta\sin(2\alpha+2\beta)}{\sin(\theta-\beta)\sin 2\alpha} \\
\Rightarrow\ & 1 &=& \frac{\sin\theta\cdot 2\cos(\alpha+\beta)}{\sin(\theta-\beta)\cdot 2\cos\alpha} \\
& && \text{(Double angle formula)} \\
\Rightarrow\ & 2\sin(\theta-\beta)\cos\alpha &=& 2\cos(\alpha+\beta)\sin\theta \\
\Rightarrow\ & \sin(\theta-\beta+\alpha)+\sin(\theta-\beta-\alpha) &=& \sin(\theta+\alpha+\beta) \\
& && \quad +\sin(\theta-\beta-\alpha) \\
& && \text{(Products to sums)} \\
\Rightarrow\ & \sin(\theta-\beta+\alpha) &=& \sin(\theta+\beta+\alpha).
\end{aligned}$$

If now $\theta-\beta+\alpha = \theta+\beta+\alpha$ then we conclude that $\beta = 0^\circ$ which is not permitted. So we must have $\theta-\beta+\alpha = 180^\circ-(\theta+\beta+\alpha)$. This simplifies to $2\alpha+2\theta = 180^\circ$. Thus $180^\circ-\theta-2\alpha = \theta$ which means that $\angle APB = \angle BAP$ so that $AB = BP$. Also $180^\circ-\theta-2\alpha-\beta = \theta-\beta$ which means that $\angle PBC = \angle PCB$ so that $BP = CP$.

Thus $AB = BP = CP$ and we are done.

RESULTS

Number of Candidates: 102

Gold

FIELD Laurence*	Sydney Grammar School	NSW	12
GUO Ivan*	Sydney Boys High School	NSW	12
NADASI Daniel	Cranbrook School	NSW	12
HUA Alex	Scotch College	VIC	12
KHAN Kamil	St Michael's Grammar School	VIC	12
COLEMAN Patrick	St Peter's College	SA	12
CHOW Sam	Scotch College	VIC	11
WAN James	Scotch College	VIC	12
WHITE Graham	James Ruse Agricultural High School	NSW	10
NGAI Andrew	James Ruse Agricultural High School	NSW	12

Silver

RAMCHEN Kim	Melbourne Grammar School	VIC	11
NANDAKUMAR Vinoth	James Ruse Agricultural High School	NSW	9
PILCH Konrad	St Peter's College	SA	10
CHAN Stephen	Knox Grammar School	NSW	12
NG Matthew	Scotch College	VIC	10
MUIRHEAD Stephen	Scotch College	VIC	11
DAI Pei	James Ruse Agricultural High School	NSW	12
KOMATSU Ray	University High School	VIC	12
ZHANG Alexander	James Ruse Agricultural High School	NSW	10
STAKER Matthew	Australian Science and Mathematics School	SA	12
LAU Yu Heng	James Ruse Agricultural High School	NSW	11
STOROZHEV Michael	Narrabundah College	ACT	11
LIBLING Matthew	Sydney Grammar School	NSW	11
SIMAKOV Nikita	Adelaide High School	SA	11
BAUSOR Michael	Shenton College	WA	12

Australian Team to participate in 2004 IMO at Athens, Greece

Laurence Field, Ivan Guo, Daniel Nadasi, Alex Hua, Kamil Khan, Graham White.

2005

Problems

PAPER 1 Tuesday 08 February

1. Let ABC be a right-angled triangle with the right angle at C. Let $BCDE$ and $ACFG$ be squares external to the triangle. Furthermore, let AE intersect BC at H, and let BG intersect AC at K.

Find the size of the angle DKH.

2. Consider a polyhedron whose faces are convex polygons.

Show that it has at least two faces with the same number of edges.

3. Let n be a positive integer, and let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 + a_2 + \cdots + a_n = n$.

Show that

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} \leq \frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}.$$

4. Show that, for each positive integer n, there exists a positive integer x such that

$$\sqrt{x + 2004^n} + \sqrt{x} = \left(\sqrt{2005} + 1\right)^n.$$

PAPER 2 Wednesday 09 February

5. In a multiple choice test there are $q \geq 10$ questions. For each question there are $a > 1$ possible answers, exactly one of which is right. A student who gets r answers right, w answers wrong and does not attempt the other questions will receive a score of

$$\frac{100(r-w)}{q(a-1)}.$$

Determine all different pairs (q, a) such that all possible scores are integers.

6. Let ABC be a triangle. Let D, E, F be points on the line segments BC, CA and AB, respectively, such that line segments AD, BE and CF meet in a single point. Suppose that $ACDF$ and $BCEF$ are cyclic quadrilaterals.

Prove that AD is perpendicular to BC, BE is perpendicular to AC, and CF is perpendicular to AB.

7. Let $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ be two sequences of integers such that $a_0 = b_0 = 1$ and, for each non-negative integer k,

(i) $a_{k+1} = b_0 + b_1 + b_2 + \cdots + b_k$ and

(ii) $b_{k+1} = \left(0^2+0+1\right)a_0 + \left(1^2+1+1\right)a_1 + \cdots + \left(k^2+k+1\right)a_k$.

For each positive integer n show that

$$a_n = \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n}.$$

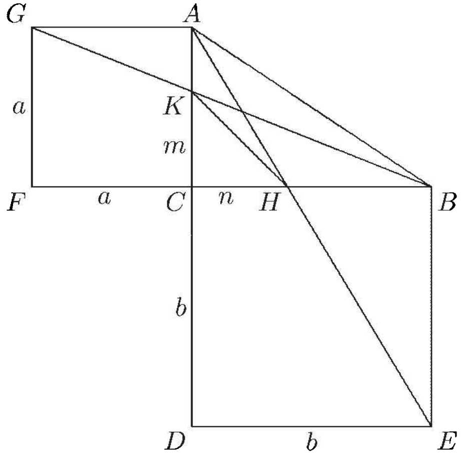
8. In an $n \times n$ array, each of n distinct symbols occurs exactly n times. An example with $n = 3$ is

1	2	3
1	3	2
2	3	1

Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct symbols.

Solution 1 (John Biun, Scotch College, VIC.)

Let the side length of squares $ACFG$ and $BCDE$ be a and b respectively. Let $CK = m$ and $CH = n$ as in the following figure.



Since $KC \parallel GF$ we see that triangles BCK and BFG are similar. Thus

$$\begin{aligned} \frac{KC}{GF} &= \frac{BC}{BF} \\ \Rightarrow \quad \frac{m}{a} &= \frac{b}{a+b} \\ \Rightarrow \quad m &= \frac{ab}{a+b}. \end{aligned}$$

Similarly since $CH \parallel DE$ we see that triangles ACH and ADE are similar. Thus

$$\begin{aligned} \frac{CH}{DE} &= \frac{AC}{AD} \\ \Rightarrow \quad \frac{n}{b} &= \frac{a}{a+b} \\ \Rightarrow \quad n &= \frac{ab}{a+b}. \end{aligned}$$

Hence we deduce that $m = n$. This means that triangle CKH is an isosceles right angled triangle. Thus $/DKH = /CKH = 45^\circ$.

Solution 2 (Jasmine Zhu, University High School, VIC)

Suppose for the sake of contradiction that there exists a polyhedron whose faces are convex but no two of its faces have the same number of edges. Let its face with the largest number of faces be called X and let X have n edges. Since there cannot be another face with n edges, all other faces must have either $n-1$, $n-2$, …, 5, 4 or 3 edges. This makes a total of $n-3$ possibilities. However since X has n edges there must be at least n other faces in the polyhedron, one for each such edge. But we have already shown that there are only $n-3$ possibilities for the number of edges of these n other faces. This means that two of these other faces must have the same number of edges - a contradiction.

Thus the original assumption is wrong and hence there are two faces with the same number of edges.

Solution 3

Alternative 1 (Raymond Chow, Belconnen High School, ACT)

For any real number x we have using the GM-HM inequality

$$\sqrt{\frac{1}{x^2}\cdot\frac{1}{1}} \geq \frac{2}{x^2+1}.$$

This may be rearranged and simplified to yield

$$\frac{x}{x^2+1} \leq \frac{1}{2}.$$

Letting x successively equal $a_1, a_2, \ldots, a_n$ and then adding the resultant inequalities shows that

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} \leq \frac{n}{2}. \qquad (*)$$

By the AM-HM inequality we have

$$\begin{aligned}
\frac{\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1}}{n} &\geq \frac{n}{(a_1+1)+(a_2+1)+\cdots+(a_n+1)} \\
&= \frac{n}{a_1+a_2+\cdots+a_n+n} \\
&= \frac{1}{2} \quad (\text{since } a_1+a_2+\cdots+a_n = n)
\end{aligned}$$

Thus

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_n+1} \geq \frac{n}{2}.$$

This together with the previous result $(*)$ completes the proof.

Alternative 2 (Charles Li, Sydney Grammar School, NSW)

For any real number x we have $(x-1)^2 \geq 0$. Thus $x^2+1 \geq 2x$ which may be rearranged to give $\frac{x}{x^2+1} \leq \frac{1}{2}$. We use this result as in the first alternative to show

$$\frac{a_1}{a_1^2+1} + \frac{a_2}{a_2^2+1} + \cdots + \frac{a_n}{a_n^2+1} \leq \frac{n}{2}. \qquad (*)$$

By the Cauchy Schwartz inequality we have

$$\begin{aligned}\left(\sum_{i=1}^{n}\frac{1}{a_i+1}\right)\left(\sum_{i=1}^{n}a_i+1\right) &\geq \left(\sum_{i=1}^{n}\frac{1}{\sqrt{a_i+1}}\cdot\sqrt{a_i+1}\right)^2\\ &= n^2.\end{aligned}$$

Since $a_1+a_2+\cdots+a_n=n$ this may be rearranged to yield

$$\frac{1}{a_1+1}+\frac{1}{a_2+1}+\cdots+\frac{1}{a_n+1}\geq\frac{n}{2}.$$

Together with $(*)$ this completes the proof.

Alternative 3 (Vinoth Nandakumar, James Ruse Agricultural High School, NSW)

To prove that $\sum_{i=1}^{n}\frac{1}{a_i+1}\geq\frac{n}{2}$ consider the function $f(x)=\frac{1}{x+1}$ for non-negative real numbers x. We compute the second derivative to be $f''(x)=\frac{2}{(x+1)^3}\geq 0$. This shows that f is convex which means that we may apply Jensen's inequality:

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n}f(a_i) &\geq f\left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)\\ \Rightarrow\quad \frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i+1} &\geq f(1)\quad\left(\text{since }\sum_{i=1}^{n}a_i=n\right)\\ &= \frac{1}{2}\\ \Rightarrow\quad \sum_{i=1}^{n}\frac{1}{a_i+1} &\geq \frac{n}{2}\end{aligned}$$

as required.

Solution 4

Alternative 1 (Graham White, James Ruse Agricultural High School, NSW)

Upon expansion using the binomial theorem we see that we may write $(\sqrt{2005}+1)^n=a_n+c_n\sqrt{2005}=a_n+b_n$ where a_n, c_n are integers and $b_n=c_n\sqrt{2005}$ so that b_n^2 is an integer. We claim that $a_n^2-b_n^2=(-2004)^n$. We prove this by induction.

For $n=1$ we have $a_1=1$ and $b_1=\sqrt{2005}$ so $a_1^2-b_1^2=1-2005=(-2004)^1$.

Now assume that the result is true for $n=k$, i.e. $a_k^2-b_k^2=(-2004)^k$. Note that

$$\begin{aligned}a_{k+1}+b_{k+1} &= (\sqrt{2005}+1)(a_k+b_k)\\ &= a_k+\sqrt{2005}b_k+b_k+\sqrt{2005}a_k.\end{aligned}$$

Since both sides have the form of an integer plus an integer times $\sqrt{2005}$, this means that we can equate the relevant parts to deduce

$$\begin{aligned} a_{k+1} &= a_k + \sqrt{2005}b_k \text{ and} \\ b_{k+1} &= b_k + \sqrt{2005}a_k. \end{aligned}$$

Hence

$$\begin{aligned} a_{k+1}^2 - b_{k+1}^2 &= a_k^2 + 2\sqrt{2005}a_k b_k + 2005b_k^2 \\ &\quad -(b_k^2 + 2\sqrt{2005}a_k b_k + 2005a_k^2) \\ &= -2004(a_k^2 - b_k^2) \\ &= -2004(-2004)^k \text{ (inductive assumption)} \\ &= (-2004)^{k+1}. \end{aligned}$$

Thus the result is true for $n = k+1$ and the induction is complete. We now show that a suitable x exists.

If n is odd set $x = a_n^2$. In this case

$$\begin{aligned} \sqrt{x} + \sqrt{x + 2004^n} &= a_n + \sqrt{a_n^2 - (a_n^2 - b_n^2)} \\ &= a_n + b_n \\ &= (\sqrt{2005} + 1)^n. \end{aligned}$$

If n is even set $x = b_n^2$. Then

$$\begin{aligned} \sqrt{x} + \sqrt{x + 2004^n} &= b_n + \sqrt{b_n^2 + (a_n^2 - b_n^2)} \\ &= b_n + a_n \\ &= (\sqrt{2005} + 1)^n. \end{aligned}$$

Since a_n^2 and b_n^2 are integers this shows that x is also an integer.

Alternative 2 (Vinh Pham, Sydney Boys High School, NSW)

We prove the following generalization: for each positive integer n, there exists a positive integer x such that

$$\sqrt{x + A^n} + \sqrt{x} = (\sqrt{A+1} + 1)^n$$

where A is a positive integer.

Proof: Let $A^n = a$ and $(\sqrt{A+1} + 1)^n = b$. We now solve by successive squaring and simplification. The equation becomes:

$$\begin{aligned} \sqrt{x + a} + \sqrt{x} &= b \\ \Rightarrow \quad x + a + x + 2\sqrt{x(x+a)} &= b^2 \\ \Rightarrow \quad 2\sqrt{(x(x+a)} &= b^2 - 2x - a \\ \Rightarrow \quad 4x(a+x) &= b^4 + 4x^2 + a^2 - 4b^2x - 2ab^2 + 4ax \\ \Rightarrow \quad 4b^2x &= b^4 - 2ab^2 + a^2 \\ &= (b^2 - a)^2 \\ \Rightarrow \quad x &= (\tfrac{1}{2}(b - \tfrac{a}{b}))^2. \end{aligned}$$

Since $(\sqrt{A+1}+1)(\sqrt{A+1}-1) = A$ we see that $\frac{a}{b} = (\frac{A}{\sqrt{A+1}+1})^n = (\sqrt{A+1}-1)^n$. Thus

$$x = \left(\frac{1}{2}\left(\left(\sqrt{A+1}+1\right)^n - \left(\sqrt{A+1}-1\right)^n\right)\right)^2 .$$

Consider $I = (\sqrt{A+1}+1)^n - (\sqrt{A+1}-1)^n$.

$$\begin{aligned} I &= \sqrt{A+1}^n + \tbinom{n}{1}\sqrt{A+1}^{n-1} + \tbinom{n}{2}\sqrt{A+1}^{n-1} + \cdots + 1 \\ &\quad -\sqrt{A+1}^n + \tbinom{n}{1}\sqrt{A+1}^{n-1} - \tbinom{n}{2}\sqrt{A+1}^{n-1} + \cdots + (-1)^n \\ &= 2\left(\tbinom{n}{1}\sqrt{A+1}^{n-1} + \tbinom{n}{3}\sqrt{A+1}^{n-3} + \cdots\right). \end{aligned}$$

If n is odd then I is an even integer which means that $x = (\frac{I}{2})^2$ is a positive integer. If on the other hand n is even then I is of the form $I = 2\sqrt{A+1}p$ where p is a positive integer. In this case $x = (\frac{I}{2})^2 = (A+1)p^2$ which is also a positive integer.

Thus in all cases x is a positive integer.

Comment: What we have shown is that *if* there is a solution to the equation then it must be the x as described above. However it is possible that the equation may not have any solutions. To show that it does we note that the function $f(x) = \sqrt{x+a^n} + \sqrt{x} - (\sqrt{a+1}+1)^n$ is continuous on the interval $[0, +\infty)$ and that $f(0) < 0$ and $\lim_{x\to\infty} f(x) = +\infty$. Hence by the intermediate value theorem $f(x) = 0$ for some positive real number x.

Solution 5 (Konrad Pilch, St Peter's College, SA)

We require $\frac{100(r-w)}{q(a-1)}$ to be an integer. Since it is possible to have $r - w = 1$ this means that $q(a-1)|100$. Conversely if $q(a-1)|100$ then $\frac{100(r-w)}{q(a-1)}$ is an integer. Hence $q|100$. Since $q \geq 10$ this means $q = 10, 20, 25, 50, 100$.

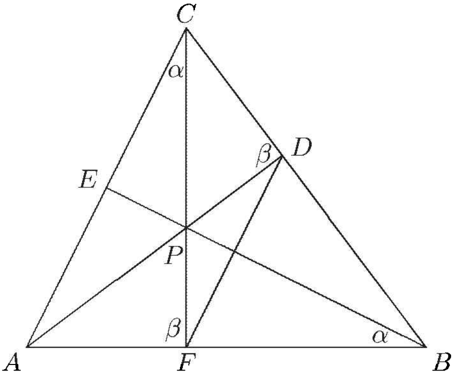
When $q = 10$ then $a - 1|10$ so $a = 2, 3, 6, 11$.
When $q = 20$ then $a - 1|5$ so $a = 2, 6$.
When $q = 25$ then $a - 1|4$ so $a = 2, 3, 5$.
When $q = 50$ then $a - 1|2$ so $a = 2, 3$.
When $q = 100$ then $a - 1|1$ so $a = 2$.

Thus in summary there are twelve solutions, namely, $(q, a) = (10, 2)$, $(10, 3)$, $(10, 6)$, $(10, 11)$, $(20, 2)$, $(20, 6)$, $(25, 2)$, $(25, 3)$, $(25, 5)$, $(50, 2)$, $(50, 3)$ and $(100, 2)$.

Solution 6

Alternative 1 (Charles Li, Sydney Grammar School, NSW)

Let AD, BE and CF all intersect at point P. Let $\angle ACF = \alpha$ and $\angle AFC = \beta$ as in the figure.



Since $BCEF$ is cyclic we have $\angle ECF = \angle EBF = \alpha$. Since $ACDF$ is cyclic we have $\angle ADF = \angle ACF = \alpha$. Thus $\angle PDF = \angle PBF = \alpha$. This means that $PDBF$ is also cyclic.

Furthermore since $ACDF$ is cyclic we have $\angle ADC = \angle AFC = \beta$. Since $PDBF$ is cyclic we have $\angle BDP = \angle AFC = \beta$. Thus $\angle BDP + \angle ADC = 2\beta = 180°$. Thus $\beta = 90°$. Thus AD and CF are altitudes. Finally from cyclic quadrilateral $BCEF$ we have $\angle BEC = \angle BFC = 90°$. Thus BE is also an altitude.

Alternative 2 (Sam Chow, Scotch College, VIC)

We refer to the same diagram as in the first alternative. Since quadrilaterals $ACDF$ and $BCEF$ are cyclic we may use Power of a Point. Thus

$$\begin{aligned} BF \cdot BA &= BD \cdot BC \\ AF \cdot AB &= AE \cdot AC. \end{aligned}$$

Dividing the first equation by the second yields

$$\frac{AF \cdot BD \cdot BC}{FB \cdot EA \cdot CA} = 1.$$

Note that we also have the following relation by Ceva's Theorem

$$\frac{AF \cdot BD \cdot CE}{FB \cdot DC \cdot EA} = 1.$$

Dividing these last two relations yields

$$BC \cdot DC = AC \cdot CE.$$

By the converse of Power of a Point this means that $AEDB$ is cyclic. We may now chase the angles around the three cyclic quadrilaterals $AEDB$,

$BCEF$ and $ACDF$ as follows

$$\begin{array}{rcll}\angle BFC & = & \angle BEC & (BCEF \text{ cyclic})\\ & = & 180^\circ - \angle AEB & \\ & = & 180^\circ - \angle ADB & (AEDB \text{ cyclic})\\ & = & \angle ADC & \\ & = & \angle AFC & (ACDF \text{ cyclic})\end{array}$$

Since $\angle BFC + \angle AFC = 180^\circ$ this means that these two angles and all of the angles in the preceding calculation are equal to 90° which solves the problem.

Solution 7

Alternative 1 (Mark Norrish, University High School, VIC)

We prove that the pair of statements

$$\begin{array}{rcl}a_k & = & k!\\ b_k & = & k\cdot k!\end{array}$$

is true for $k \geq 1$ by induction.

Note that $a_1 = b_0 = 1$ and $b_1 = a_0 = 1$ so that the pair of statements is true for $k = 1$.

Let us assume that the pair of statements is true for k. We prove that the pair of statements is also true for $k+1$. Note that

$$\begin{array}{rcl}a_{k+1} & = & (b_0 + b_1 + \cdots + b_{k-1}) + b_k\\ & = & a_k + b_k\\ & = & k! + k\cdot k! \quad \text{(by inductive assumption)}\\ & = & (k+1)!\end{array}$$

and

$$\begin{array}{rcl}b_{k+1} & = & ((0^2+0+1)a_0 + (1^2+1+1)a_1 + \cdots\\ & & \quad +((k-1)^2 + (k-1) + 1)a_{k-1}) + (k^2+k+1)a_k\\ & = & b_k + (k^2+k+1)a_k\\ & = & k\cdot k! + (k^2+k+1)k! \quad \text{(by inductive assumption)}\\ & = & (k^2+2k+1)k!\\ & = & (k+1)\cdot(k+1)!\end{array}$$

Thus the pair of statements is also true for $k+1$. Hence the formulas for a_k and b_k are true for all $k \geq 1$. It follows that

$$\begin{array}{rcl}\dfrac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n} & = & \dfrac{1\cdot 1!\cdot 2\cdot 2!\cdot\ldots\cdot n\cdot n!}{1!2!\ldots n!}\\ & = & n!\end{array}$$

as required.

Alternative 2 (Angus McInnes, De La Salle College, VIC)

We prove that $b_k = ka_k$ for all positive integers k by induction.

Note that it is true for $k = 1$.

Assume now that it is true for k. As in the first alternative note that

$$\begin{aligned} a_{k+1} &= a_k + b_k \\ &= a_k + ka_k \quad \text{(by inductive assumption)} \\ &= (k+1)a_k \quad (*) \end{aligned}$$

and as in the first alternative

$$\begin{aligned} b_{k+1} &= b_k + (k^2 + k + 1)a_k \\ &= ka_k + (k^2 + k + 1)a_k \quad \text{(by inductive assumption)} \\ &= (k+1)^2 a_k \\ &= (k+1)a_{k+1} \quad \text{(from } (*)\text{)}. \end{aligned}$$

Thus the induction is complete. Hence $\frac{b_k}{a_k} = k$ for all positive integers k. Hence

$$\begin{aligned} \frac{b_1 b_2 \ldots b_n}{a_1 a_2 \ldots a_n} &= \frac{b_1}{a_1} \cdot \frac{b_2}{a_2} \cdot \ldots \cdot \frac{b_n}{a_n} \\ &= n! \end{aligned}$$

Solution 8 (Sam Chow, Scotch College, VIC)

Let h_m be the number of distinct symbols in the mth row and let g_m be the number of distinct symbols in the mth column. Let x_k be the number of rows which k occurs in and let y_k be the number of columns which k occurs in.

Suppose for the sake of contradiction that $h_m, g_m < \sqrt{n}$ for $m = 1, 2, \ldots,$ n. Since the symbol k occurs n times in total and it occurs in x_k rows then it follows that it must occur at least $\frac{n}{x_k}$ times in some row. Furthermore each square of that row belongs to a different column, thus k must occur in at least $\frac{n}{x_k}$ columns. This means that $y_k \geq \frac{n}{x_k}$. Thus $\sqrt{x_k y_k} \geq \sqrt{n}$. Hence from our assumption above we have the following inequalities

$$\sqrt{x_k y_k} > h_m \tag{1}$$

and

$$\sqrt{x_k y_k} > g_m. \tag{2}$$

Consider $\sum_{m=1}^{n} h_m$. Each symbol is counted once for every row it is in. Thus

$$\sum_{m=1}^{n} h_m = \sum_{k=1}^{n} x_k.$$

Thus using (1) we derive

$$\sum_{k=1}^{n} \sqrt{x_k y_k} > \sum_{k=1}^{n} x_k. \tag{3}$$

Similarly

$$\sum_{k=1}^{n} \sqrt{x_k y_k} > \sum_{k=1}^{n} y_k. \tag{4}$$

Multiplying (3) and (4) together yields

$$\left(\sum_{k=1}^{n} \sqrt{x_k y_k}\right)^2 > \sum_{k=1}^{n} x_k \sum_{k=1}^{n} y_k.$$

However this is in direct conflict with the Cauchy Schwarz Inequality which states

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2.$$

Indeed setting $a_k = \sqrt{x_k}$ and $b_k = \sqrt{y_k}$ yields the desired contradiction. Thus there is a row or column with at least $\sqrt{n}$ distinct symbols.

RESULTS

Number of Candidates: 106

Gold

CHOW Sam	Scotch College	VIC	12
PHAM Vinh	Sydney Boys High School	NSW	11
LI Charles	Sydney Grammar School	NSW	11
PILCH Konrad	St Peter's College	SA	11
NANDAKUMAR Vinoth	James Ruse Agricultural High School	NSW	10
RAMCHEN Kim	Melbourne Grammar School	VIC	12
MUIRHEAD Stephen	Scotch College	VIC	12
LIBLING Matt	Sydney Grammar School	NSW	12
YAN Harry	James Ruse Agricultural High School	NSW	11
MORRIS Anthony	Sydney Boys High School	NSW	10
WHITE Graham	James Ruse Agricultural High School	NSW	11

Silver

NG Matthew	Scotch College	VIC	11
LEONG Chris	James Ruse Agricultural High School	NSW	12
LEUNG Brian	Sydney Grammar School	NSW	12
TANG Howard	Brisbane Grammar School	QLD	12
ZHU Jasmine	University High School	VIC	11
CHUANG Howard	Brisbane State High School	QLD	12
ZHANG Alexander	James Ruse Agricultural High School	NSW	11
LIN Andrew	Anglican Church Grammar School	QLD	12
STOROZHEV Michael	Narrabundah College	ACT	12
HA Saleem Shing Kwan	James Ruse Agricultural High School	NSW	11
SHAH Margit	James Ruse Agricultural High School	NSW	11

McINNES Angus	De La Salle College	VIC	10
YIN Albert	James Ruse Agricultural High School	NSW	11
LAU Yu Heng	James Ruse Agricultural High School	NSW	12
CHOW Raymond	Belconnen High School	ACT	10
LUC Alan	James Ruse Agricultural High School	NSW	11
RANASINGHE Navin	Scotch College	VIC	11
GOUNDER Praveen	James Ruse Agricultural High School	NSW	12
TALANINE Kirill	Canberra Grammar School	ACT	12

Australian Team to participate in 2005 IMO at Merida, Mexico

Sam Chow, Konrad Pilch, Vinoth Nandakumar, Kim Ramchen, Stephen Muirhead, Graham White.

2006

Problems

PAPER 1 Tuesday 07 February

1. Find all solutions to $1 + 5 \times 2^m = n^2$, where m and n are positive integers.

2. Let f be a function taking positive integers to positive integers such that

(i) $f(ab) = f(a)f(b)$;

(ii) $f(a) < f(b)$ whenever $a < b$;

(iii) $f(3) \geq 7$.

Find the smallest value that $f(3)$ can take.

3. Let $PRUS$ be a trapezium with PR parallel to SU such that $/SPU = 2/UPR$ and $/PSR = 2/RSU$. Suppose Q is on PR such that QS bisects $/PSR$ and, similarly, T is on SU such that TP bisects $/SPU$. Let PT meet SQ at E, and let PU meet SR at F. The line through E parallel to SR meets PU at G, and the line through E parallel to PU meets SR at H. Finally, the line through G and H meets PR at K and SU at L.

Prove that $KG = GH = HL$.

4. Let $P_1, P_2, \ldots, P_n$ be n different points on a circle. Between each pair of points there is a line segment which is coloured either red or blue. Consider colourings for which P_iP_j is red if and only if $P_{i+1}P_{j+1}$ is blue, for any distinct i and j in the set $\{1, \ldots, n\}$. We interpret P_{n+1} as being the same point as P_1.

(a) For which values of n is such a colouring possible?

(b) Let a *step* consist of moving along a single red segment from one point to another point.

Show that it is possible to get from each point to any other point in at most three steps.

PAPER 2 Wednesday 08 February

5. Let $ABCD$ be a square, and let E be a point on its diagonal BD. Suppose that O_1 is the centre of the circle passing through ABE and O_2 is the centre of the circle passing through ADE.

Show that AO_1EO_2 is a square.

6. For each positive integer n, let $a(n)$ denote the product of all digits of n.

(a) Show that $a(n) \leq n$.

(b) Find all solutions to the equation $n^2 - 17n + 56 = a(n)$.

7. For each sequence $S = (a_1, a_2, \ldots, a_n)$ of non-negative integers let the *offspring* of S be the sequence $T = (b_1, b_2, \ldots, b_n)$, where b_i is the number of integers in S to the right of a_i that are less than a_i. For example,

(a) if $S = (6, 1, 8, 0, 5, 7, 2, 2, 4, 0, 7, 7, 5)$,

(b) then $T = (8, 2, 10, 0, 4, 5, 1, 1, 1, 0, 1, 1, 0)$.

For a given sequence S_0, let S_1 be the offspring of S_0, S_2 the offspring of S_1, and so on.

Show that there exists an integer j such that $S_j = S_{j+1}$.

8. Alice and Bob play the following guessing game. Alice chooses a positive integer a and tells Bob that it is at most 2006. At each turn, Bob chooses a positive integer b and calls it out to Alice. Alice then tells Bob whether or not $a + b$ is a prime.

Prove that there is a strategy for Bob to determine a in less than 2006 turns.

Solution 1

Alternative 1 (Konrad Pilch, St Peter's College, SA)

We rewrite the equation as

$$(n-1)(n+1) = 5 \times 2^m.$$

We notice that $n-1$ and $n+1$ have the same parity. As m is positive, at least one of $n-1$, $n+1$ is even. Hence they are both even. Let $n-1 = 2a$ and so $n+1 = 2a+2$. The equation becomes

$$a(a+1) = 5 \times 2^{m-2}.$$

Now a, $a+1$ are of different parity. Hence one of them is odd. The only possibilities for the odd one are 1 or 5. If one of them equals 1, the other equals $5 \times 2^{m-2} \geq 5$ so that a, $a+1$ could not possibly be consecutive integers. Hence the odd one equals 5. If $a = 5$ then $2^{m-2} = 6$ which is impossible. If $a+1 = 5$ then $2^{m-2} = 4$ which leads to $m = 4$ and $n = 9$ which can easily be verified as solving the equation. Hence the only solution is $m = 4$ and $n = 9$.

Alternative 2 (May Luo, St George Girls High School, NSW)

Since $m \geq 1$ we note that the last digit of $1 + 2^m \times 5$ is 1. Since the last digit of n^2 equals the square of the last digit of n we deduce (by checking the digits $0, 1, \ldots, 9$) that the last digit of n must be 1 or 9. Thus there are two cases. Either $n = 10x + 1$ or $n = 10x + 9$ where $x \geq 0$ in each case.

Upon substitution and simplification we find that

$$2^{m-2} = x(5x+1) \text{ or } (x+1)(5x+4).$$

Note that in both cases the factors on the right hand side have opposite parity because their respective differences ($(5x+1) - x = 4x+1$ or $(5x+4) - (x+1) = 4x+3$) are odd. Yet both factors must be powers of 2. Since the only odd power of 2 is 1, this means that the smaller factor in each instance must equal 1. In the first case this implies that $x = 0$ or $x = 1$ neither of which result in $x(5x+1)$ being a power of 2. In the second case it means that $x = 0$ in which case $(x+1)(5x+4) = 2^2$. Hence $m = 4$ and $n = 10x + 9 = 9$. Substitution verifies that $(m, n) = (4, 9)$ is indeed a solution.

Alternative 3 (Navin Ranasinghe, Scotch College, VIC)

We rewrite the equation as

$$5 \times 2^m = (n+1)(n-1).$$

Thus one factor on the right hand side equals 5×2^x and the other equals 2^{m-x} where $0 \leq x \leq m$. Since $(n+1)-(n-1)=2$ this means that

$$|5 \times 2^x - 2^{m-x}| = 2$$

and so

$$|5 - 2^{m-2x}| = 2^{1-x}.$$

Suppose $m - 2x \leq 0$, then $4 \leq |5 - 2^{m-2x}| < 5$ while $0 < 2^{1-x} \leq 2$ which is a contradiction. Thus $|5 - 2^{m-2x}|$ is an odd positive integer. The only power of 2 which is an odd positive integer is when $2^{1-x} = 2^0$. Thus $x = 1$ which in turn implies $|5-2^{m-2}| = 1$. Hence either $2^{m-2} = 6$ which is impossible or $2^{m-2} = 4$ which leads to $m = 4$ and then $n = 9$.

Solution 2

Alternative 1 (Jasmine Zhu, University High School, VIC)

Firstly we note that $f(x) = x^2$ is a solution that satisfies all the conditions with $f(3) = 9$. We claim that 9 is the smallest value that $f(3)$ can take.

From (i) and (ii) we know $f(2)f(5) = f(10) > f(9) = f(3)f(3)$ and $f(2)f(2)f(2)f(2) = f(4)f(4) = f(16) > f(15) = f(3)f(5)$. Hence the following two inequalities hold:

$$\begin{aligned} f(2)f(5) &> f(3)^2 \\ f(2)^4 &> f(3)f(5). \end{aligned}$$

Multiplying the two inequalities together yields

$$\begin{array}{rrcll} & f(2)^5 f(5) & > & f(3)^3 f(5) & \\ & & \geq & 7^3 f(5) & \text{(since } f(3) \geq 7) \\ \Rightarrow & f(2) & \geq & \sqrt[5]{343} & \\ \Rightarrow & f(2) & \geq & 4 & \text{(since } f(2) \text{ is an integer).} \end{array}$$

But then since $f(9) > f(8)$, this means that $f(3)^2 > f(2)^3 \geq 4^3$. Hence $f(3) > 8$. Thus $f(3) \geq 9$. This completes the proof.

Alternative 2 (Konrad Pilch, St Peter's College, SA)

It can be seen that $f(3) = 9$ is possible since the function $f(x) = x^2$ satisfies the three criteria. We now show that $f(3) \geq 9$.

By condition (i) we know $f(b) = f(1)f(b)$. Hence $f(1) = 1$. Condition (ii) implies that $f(2) \geq 2$.

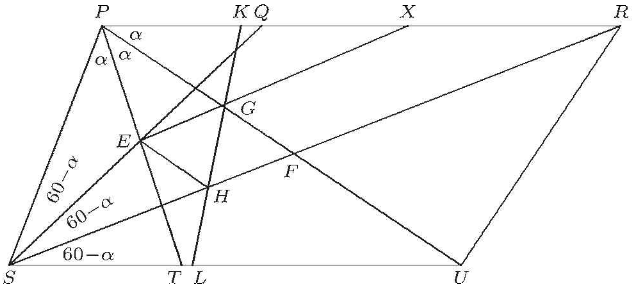
Suppose that $f(2) = 2$. Then $f(4) = f(2)^2 = 4$. However $f(3) < f(4) = 4$ by condition (ii). However this violates condition (iii).

Suppose that $f(2) = 3$. Then $f(27) = f(3)^3 \geq 7^3 = 343$ and $f(32) = f(2)^5 = 3^5 = 243$. So $f(27) > f(32)$ which violates condition (ii).

Hence $f(2) \geq 4$. This implies that $f(8) = f(2)^3 \geq 64$. Then also $f(3)^2 = f(9) > f(8) \geq 64$. Hence $f(3) > 8$. This means that $f(3) \geq 9$, as required.

Solution 3

Alternative 1 (Matthew Ng, Scotch College, VIC)



Let $\angle SPT = \angle TPU = \angle UPR = \alpha$. Then $\angle PSU = 180° - \angle SPR = 180° - 3\alpha$. Hence angle $\angle PSQ = \angle QSR = \angle RSU = 60° - \alpha$. Thus using the angle sum in triangle PSF we see that

$$\begin{aligned} \angle PFS &= 180° - \angle FPS - \angle PSF \\ &= 180° - 2\alpha - 2(60° - \alpha) \\ &= 60°. \end{aligned}$$

Note that PE bisects $\angle FPS$ and SE bisects $\angle FSP$. This implies that E is the incentre of triangle FSP. Hence FE bisects $\angle PFS$, that is FE bisects $\angle GFH$.

Consider the quadrilateral $EGFH$. It is a parallelogram since it is given that $GF \parallel EH$ and $EG \parallel HF$. Yet we deduced that its diagonal FE is an angle bisector of $\angle GFH$. Thus $\angle GFE = \angle HFE = \angle GEF$ (from parallel lines). Hence triangle GEF is isosceles with $GE = GF$. This means that $EGFH$ is in fact a rhombus with $\angle GFH = \angle PFS = 60°$. Hence GH bisects $\angle EGF$ and $\angle EGF = 120°$. Hence $\angle EGH = \angle FGH = 60°$. Hence triangle HGF is equilateral, as is also triangle EGH.

We now identify two congruent triangles, namely triangles PGK and PGE. This is true because $\angle PGK = \angle FGH = 60° = \angle GFH = \angle PGE$ (since $EG \parallel HF$), $\angle GPK = \alpha = \angle GPE$ and GP is a common side.

Thus $KG = GE$. Similarly $HL = HE$. However since triangle EGH is equilateral this means that $GE = HE = GH$. Hence $KG = GH = HL$.

Alternative 2 (Graham White, James Ruse Agricultural High School, NSW)

We use the notation as in the first alternative, namely $\angle SPT = \angle TPU = \angle UPR = \alpha$ and then also $\angle PSQ = \angle QSR = \angle RSU = 60^\circ - \alpha$. Extend EG to meet PR at X. Then $\angle PGE = \angle XPG + \angle GXP = \angle RPU + \angle SRP = \alpha + \angle RSU = \alpha + 60^\circ - \alpha = 60^\circ$. Also $\angle PEG = 180^\circ - \angle EPG - \angle PGE = 120^\circ - \alpha$. Furthermore $\angle PES = 180^\circ - \angle SPE - \angle ESP = 180^\circ - \alpha - (60^\circ - \alpha) = 120^\circ$ and $\angle PUS = \angle UPR = \alpha$ (parallel lines).

The sine rule applied once each in triangles PGE, PSE, PSU yields respectively

$$\frac{PE}{PG} = \frac{\sin 60^\circ}{\sin(120^\circ - \alpha)}$$

$$\frac{PS}{PE} = \frac{\sin 120^\circ}{\sin(60^\circ - \alpha)}$$

$$\frac{PU}{PS} = \frac{\sin(180^\circ - 3\alpha)}{\sin \alpha}.$$

Multiplying these three results together yields

$$\begin{aligned}
\frac{PU}{PG} &= \frac{\frac{3}{4}\sin(180^\circ - 3\alpha)}{\sin\alpha \sin(120^\circ - \alpha)\sin(60^\circ - \alpha)} \\
&= \frac{3\sin 3\alpha}{4\sin\alpha\sin(60^\circ + \alpha)\sin(60^\circ - \alpha)} \\
&= \frac{3(3\sin\alpha - 4\sin^3\alpha)}{4\sin\alpha\sin(60^\circ + \alpha)\sin(60^\circ - \alpha)} \quad \text{(triple angle formula)} \\
&= \frac{3(3 - 4\sin^2\alpha)}{4(\sin 60^\circ\cos\alpha + \sin\alpha\cos 60^\circ)(\sin 60^\circ\cos\alpha - \sin\alpha\cos 60^\circ)} \\
&= \frac{3(3 - 4\sin^2\alpha)}{4(\sin^2 60^\circ\cos^2\alpha - \sin^2\alpha\cos^2 60^\circ)} \\
&= \frac{3(3 - 4\sin^2\alpha)}{3\cos^2\alpha - \sin^2\alpha} \\
&= \frac{3(3 - 4\sin^2\alpha)}{3(1 - \sin^2\alpha) - \sin^2\alpha} \\
&= \frac{3(3 - 4\sin^2\alpha)}{3 - 4\sin^2\alpha} \\
&= 3.
\end{aligned}$$

This means that $GP = \frac{1}{3}PU$ and so $GU = \frac{2}{3}PU$. Hence $GP = \frac{1}{2}GU$. Consider triangles PKG and ULG. They are similar because $PK \parallel LU$. Thus

$$\frac{KG}{GL} = \frac{GP}{GU} = \frac{1}{2}.$$

Hence $KG = \frac{1}{3}KL$. Similarly $LH = \frac{1}{3}LK$. Thus KG and HL and hence also GH are all equal to $\frac{1}{3}KL$. So $KG = GH = HL$ as required.

Solution 4 (Matthew Ng, Scotch College, VIC)

(a) Suppose P_iP_j is red. Then $P_{i+1}P_{j+1}$ is blue and then $P_{i+2}P_{j+2}$ is red and again then $P_{i+3}P_{j+3}$ is blue. We may continue this inductively to deduce that $P_{i+2k}P_{j+2k}$ is red and $P_{i+2k+1}P_{j+2k+1}$ is blue for any positive integer k where the indices are considered modulo n. But we know that $P_{i+n}P_{j+n} = P_iP_j$ is red. Hence $i + n = i + 2k$ for some positive integer k. Hence n is even.

Let $P_iP_{i+\frac{n}{2}}$ be red for some i. Then $P_{i+\frac{n}{2}}P_{(i+\frac{n}{2})+\frac{n}{2}}$ is also red because $P_iP_{i+\frac{n}{2}} = P_{i+n}P_{i+\frac{n}{2}}$. Thus from our inductive result above, this means that $\frac{n}{2}$ is even.

Since n and $\frac{n}{2}$ are both even, this implies that n is divisible by 4. We claim that all n such that $4|n$ are possible. We may construct a colouring as follows: Let all lines P_iP_{i+1}, P_iP_{i+2}, …, $P_iP_{i+\frac{n}{2}}$ be red if $2|i$ and blue if $2\not|i$. This construction satisfies P_iP_j is red if and only if $P_{i+1}P_{j+1}$ is blue and all lines are covered exactly once except for $P_iP_{i+\frac{n}{2}}$. Since $\frac{n}{2}$ is even we see that there is no conflict in the colouring because i and $i + \frac{n}{2}$ will have the same parity so that $P_iP_{i+\frac{n}{2}}$ and $P_{i+\frac{n}{2}}P_{i+\frac{n}{2}+\frac{n}{2}} = P_{i+\frac{n}{2}}P_i$ receive the same colouring.

(b) Consider two points P_i and P_j. If P_iP_j is already red then we are done. Assume henceforth that P_iP_j is blue.

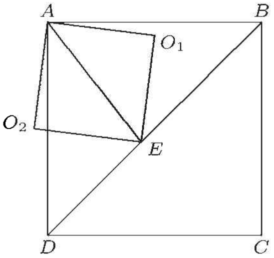
Suppose $i - j$ is even. Consider the lines P_iP_{i+1} and $P_{i-1}P_i$. We know that one of them is red. Since $i - j$ is even, the pair of lines (P_iP_{i+1}, P_jP_{j+1}) are the same colour, and they are opposite in colour to the pair of lines $(P_{i-1}P_i, P_{j-1}P_j)$. Since P_iP_j is blue, then we also know that both $P_{i-1}P_{j-1}$ and $P_{i+1}P_{j+1}$ are red. Hence one of the three-step paths $P_iP_{i+1}P_{j+1}P_j$ or $P_iP_{i-1}P_{j-1}P_j$ is entirely red.

Suppose that $i - j$ is odd. Once again one of the lines P_iP_{i+1}, $P_{i-1}P_i$ is red. Furthermore since $i - j$ is odd then $i - j + 1$ and $i - j - 1$ are both even and so the pair of lines $(P_{i-1}P_i, P_jP_{j+1})$ have the same colour, and they are opposite in colour to the pair of lines $(P_iP_{i+1}, P_{j-1}P_j)$. If $(P_{i-1}P_i, P_jP_{j+1})$ are red, then since we know that one of $P_{i-1}P_j$ and P_iP_{j+1} is red, then one of the two-step paths $P_iP_{i-1}P_j$ or $P_iP_{j+1}P_j$ is entirely red. If on the other hand $(P_iP_{i+1}, P_{j-1}P_j)$ are both red, then since we know that one of P_iP_{j-1} or $P_{i+1}P_j$ is red, then one of the two-step paths $P_iP_{j-1}P_j$ or $P_iP_{i+1}P_j$ is entirely red.

Thus in all cases we have a path which takes us to any other point in at most three steps.

Solution 5

Alternative 1 (John Biun, Scotch College, VIC)



Note that $\angle ADE = \angle ADB = 45°$ since AB is the diagonal of a square. Since the angle subtended at the centre of a circle equals twice the angle subtended at the circumference we have $\angle AO_2E = 2\angle ADE = 90°$. Furthermore since O_2 is the circumcentre of triangle ADE, this means that triangle AO_2E is isosceles with $O_2A = O_2E$. Thus $\angle O_2AE = \angle O_2EA = 45°$. By similar reasoning since O_1 is the circumcentre of triangle ABE we find that $\angle AO_1E = 90°$ and $\angle O_1AE = \angle O_2AE = 45°$. Thus AO_1EO_2 has all angles equal to $90°$. Furthermore triangles AO_2E and AO_1E are congruent. (ASA) Therefore AO_1EO_2 has all its lengths equal. Since all the sides of AO_1EO_2 are equal and all the angles equal $90°$ this means that AO_1EO_2 is a square.

Alternative 2 (Sean Gomes, St Aloysius College, NSW)

As in *Alternative 1* we prove that $\angle AO_1E = 90° = \angle AO_2E$. Let $AO_1 = O_1E = x$ and $AO_2 = O_2E = y$. Then by Pythagoras' Theorem in right angled triangles AO_1E and AO_2E we have $2x^2 = AE^2 = 2y^2$. Thus $x = y$. This means that AO_1EO_2 is a rhombus. However since $\angle AO_1E = 90°$ this means that AO_1EO_2 is in fact a square.

Solution 6

Alternative 1 (Charles Li, Sydney Grammar School, NSW)

(a) Suppose n has k $(k \geq 1)$ digits $b_k, b_{k-1}, \ldots, b_1$ so that

$$n = b_kb_{k-1}\ldots b_1$$

is the decimal representation of n. Then

$$\begin{aligned} a(n) &= b_k \cdot b_{k-1} \cdot \ldots \cdot b_1 \\ &\leq b_k \cdot \underbrace{9 \cdot \ldots \cdot 9}_{k-1} \quad (\text{since } b_i \leq 9) \\ &= 9^{k-1} b_k. \end{aligned}$$

However $n = b_1 + 10b_2 + \cdots + 10^{k-1}b_k \geq 10^{k-1}b_k \geq 9^{k-1}b_k \geq a(n)$ (as $k \geq 1$). So $n \geq a(n)$.

(b) From (a) we know $a(n) \leq n$. But all the digits of n are non-negative. Hence $0 \leq a(n)$. As $n^2 - 17n + 56 = a(n)$ then

$$0 \leq n^2 - 17n + 56 \leq n.$$

Consider first when $n^2 - 17n + 56 \geq 0$. By solving $n^2 - 17n + 56 = 0$ we find that the quadratic expression $n^2 - 17n + 56$ factors as

$$(n - \frac{1}{2}(17 - \sqrt{65}))(n - \frac{1}{2}(17 + \sqrt{65})).$$

Since $n^2 - 17n + 56$ is a positive quadratic this means that if $n^2 - 17n + 56 \geq 0$ then

$$n \geq \frac{1}{2}(17 + \sqrt{65}) > \frac{1}{2}(17 + \sqrt{64}) = 12\frac{1}{2}$$

or

$$n \leq \frac{1}{2}(17 - \sqrt{65}) < \frac{1}{2}(17 - \sqrt{64}) = 4\frac{1}{2}.$$

But n is an integer so $n \geq 13$ or $n \leq 4$.

Now consider when $n^2 - 17n + 56 \leq n$. This easily rearranges and factorises as $(n - 4)(n - 14) \leq 0$. Thus $4 \leq n \leq 14$.

Combining the two restrictions on n we see that only $n = 4$ or $n = 13$ or $n = 14$ remain as possibilities. Checking these cases individually we see that only $n = 4$ is a solution.

Alternative 2 (Giles Gardam, Hurlstone Agricultural High School, NSW)

(a) See *Alternative 1.*

(b) First consider when n is a one-digit number. Then $a(n) = n$. We solve $n^2 - 17n + 56 = n$ to find that $n = 4$ or $n = 14$. Therefore $n = 4$ is the only one-digit solution.

We now look for solutions with more than one digit. Using the inequality from part (a) we know $n^2 - 17n + 56 \leq n$. This rearranges as $(n-4)(n-14) \leq 0$. Thus $4 \leq n \leq 14$. We already know that $n = 4$ is the only one-digit solution. Hence it remains only to test $n = 10, 11, 12, 13$ and 14. Checking each of these cases individually we see that there is no further solution.

Solution 7

Comment: In this question it was very easy to make a number of incorrect assumptions about the structure of sequences and their offsprings. The cause of this seemed to be based on experimenting with a small number of examples and quickly assuming that they were representative for any situation. For instance many students thought that all sequences would eventually freeze at a sequence which apart from inserted zeros could not increase at any point. Examples such as 7770040110 were given.

Another incorrect assumption was that the frozen right hand part of a sequence would always have to grow by at least one to the left with each successive offspring. A counter example to both claims is the following sequence and its successive offsprings: 10343110, 20343110, 40343110, 60343110, 70343110.

Alternative 1 (Anthony Morris, Sydney Boys High School, NSW)

We prove by induction that the subsequence $(a_{n-k+1}, a_{n-k+2}, \ldots, a_n)$ of k numbers eventually freezes for $k = 1, 2, \ldots n$.

For $k = 1$ it is obvious that the 1-term subsequence (a_n) freezes at $a_n = 0$ from S_1 since there are no terms to the right of a_n.

Assume that the subsequence $(a_{n-k+1}, a_{n-k+2}, \ldots, a_n)$ of k numbers freezes from some point onwards. We will prove that a_{n-k} will also eventually freeze.

Now a_{n-k} in the next offspring is equal to the number of numbers in the subsequence $(a_{n-k+1}, a_{n-k+2}, \ldots, a_n)$ of the current offspring which are less than it. If the value of a_{n-k} in the next offspring equals the value of a_{n-k} in the current offspring then a_{n-k} has already frozen. If the value of a_{n-k} in the next offspring is higher than the value of a_{n-k} in the current offspring then it must be at least as high again in all subsequent offsprings because there will be at least as many numbers to the right of it which are less than it. Then since a_{n-k} is bounded above by k (the total number of terms to the right of it) we see that a_{n-k} can only increase finitely many times until it reaches a maximum and freezes.

Similarly if the value of a_{n-k} decreases in the next offspring then it must be at most as high again in all subsequent offsprings because there will be at most as many numbers to the right of it which are less than it. Then since a_{n-k} is bounded below by 0 we see that a_{n-k} can only decrease finitely many times until it reaches a minimum and freezes.

Thus in all cases a_{n-k} eventually freezes and hence the subsequence $(a_{n-k}, a_{n-k+1}, \ldots, a_n)$ eventually freezes.

Thus the induction is complete. Finally we invoke the case $k = n$ from the induction which completes the proof.

Alternative 2 (Jet Holloway, The Hutchins School, TAS)

As a preliminary observation we note that a_i only has $n - i$ terms to the right of it, hence from S_1 onwards we must have $a_i \leq n - i$ for $i = 1, 2, \ldots, n$.

We shall now prove that from S_1 onwards that no term can decrease. That is if $S_k = (a_1, a_2, \ldots, a_n)$ and $S_{k+1} = (b_1, b_2, \ldots, b_n)$ where $k \geq 1$, then $b_i \geq a_i$ for $i = 1, 2, \ldots, n$.

Let $a_i = x$. Then from our first observation $x = a_i \leq n - i$. Therefore since x, i, n are all integers we have $i \leq n - x < n - x + 1$. Hence a_i is to the left of $a_{n-x+1}, a_{x-n+2}, \ldots, a_n$. Note that there are x numbers in this list. Furthermore from our first observation we have $a_n \leq 0$, $a_{n-1} \leq 1$, …, $a_{n-x+1} \leq x - 1$. Thus it is guaranteed that at least x numbers to the right of a_i are less than x. Hence $b_i \geq x = a_i$.

To finish the proof we note that because a_i has an upper limit of $n - i$, the a_i cannot go up forever. Thus at some stage all of the numbers must eventually stop and it is at this stage that the sequence will equal its offspring.

Alternative 3 (Angelo Di Pasquale)

It is possible to prove that if $S_0 = (a_1, a_2, \ldots, a_n)$ then $S_n = S_{n+1}$. The reason is a simple consequence of the following result, namely that after the kth offspring (i.e. in S_k) all the 0s, 1s, …, $(k-1)$s are frozen and that no new 0s, 1s, …, $(k-1)$s are formed afterwards. This claim can be established by induction. Indeed it is trivial for $k = 1$ that the 0s are frozen in S_1. Assume that it is true for $k = m$ so that all the 0s, 1s, …, $(m-1)$s are frozen. Then consider any element in S_{m+1} which equals m. The only reason that m could change in S_{m+2} is if the number of elements to its right which are less than or equal to $m - 1$ changed in going from S_m to S_{m+1}. However the inductive assumption precludes this from happening.

Having established this result we now only need to note that from S_1 onwards we have $a_i \leq n - 1$ for $i = 1, 2, \ldots, n$ because each element has no more than $n - 1$ elements on its right. We now use the result above for the case $k = n$ which yields that $S_n = S_{n+1}$.

Solution 8

Alternative 1 (Charles Li, Sydney Grammar School, NSW)

It is sufficient for Bob to eliminate one possible value for a per turn. Then even after 2005 turns, he will have eliminated 2005 possible values leaving only one possibility for a.

Consider the following sequence of numbers $2006! + 2$, $2006! + 3$, …, $2006! + 2006$. We can see that these 2005 numbers are all composite because the first term is divisible by 2, the second by 3 and so on.

Let p be the largest prime less than $2006! + 2$. (Note that p is very much larger than 2006.) The next prime larger than p is at least $2006! + 2007 \geq p + 2006$. The point is that p is prime and $p + 1, p + 2, \ldots, p + 2005$ are all composite.

Now on to Bob. For turn 1, Bob chooses $b = p - 1$. If $a + b$ is prime then since $a + b \leq p + 2005$ then would mean that $a + b = p$ so that $a = 1$. If $a + b$ is not prime then $a \geq 1$. Hence either we are done, or we have eliminated one possibility.

For turn 2, Bob chooses $b = p - 2$ and decides whether or not $a = 2$ or $a > 2$. In general on turn k, Bob has already eliminated $a = 1, 2, \ldots, k - 1$ so that $a \geq k$ so he chooses $b = p - k$. Thus $p \leq a + b \leq p + 2005$. If $a + b$ is prime then $a = k$ otherwise $a > k$.

If Bob determines a before 2005 turns then he has succeeded. If after 2005 turns Bob has not determined a then Bob knows that $a = 2006$ so he has succeeded in this case also.

Alternative 2 (Graham White, James Ruse Agricultural High School, NSW)

Bob keeps a list of possible values of a. Initially this list consists of the numbers $1, 2, \ldots, 2006$. On each turn we prove that Bob can eliminate one number from his list.

Suppose that a_1 and a_2 are in Bob's list. It suffices to prove that Bob can choose a positive integer b so that exactly one of $a_1 + b$ or $a_2 + b$ is prime. In this way he can eliminate one of a_1 or a_2.

Define the function $p(n) = 1$ if n is prime and $p(n) = 0$ otherwise. It suffices to show that for all $a_2 > a_1$ there exists b such that $p(a_1 + b) \neq p(a_2 + b)$. Assume this is false. This means that $p(x) = p(x + d)$ for all $x > a_1$ where $d = a_2 - a_1$. Hence p is periodic from a_1 onwards with period at most d.

Consider the numbers $n! + 2$, $n! + 3$, …, $n! + n$. They are all composite as $i | n! + i$ for $i = 2, 3, \ldots, n$. In particular if we let $n = d + 1$ then we have d consecutive composite numbers, i.e. $p(x) = p(x + 1) = \cdots = p(x + d - 1) = 0$ where $x = (d + 1)! + 2$. Since p has period d this means that p must be the constant function $p(x) \equiv 0$ for all $x \geq (d+1)! + 2$. This means that all integers greater than x are composite. This contradicts that there are infinitely many primes.

Thus Bob can eliminate one possibility on each turn. Hence Bob can eliminate 2005 possibilities in 2005 turns. Once he has done this he only has a single number left on his list and this is Alice's number.

RESULTS

Number of Candidates: 100

Gold

WHITE Graham*	James Ruse Agricultural High School	NSW	12
LI Charles	Sydney Grammar School	NSW	12
NG Matthew	Scotch College	VIC	12
NANDAKUMAR Vinoth	James Ruse Agricultural High School	NSW	11
PILCH Konrad	St Peter's College	SA	12
PHAM Vinh	Sydney Boys High School	NSW	12
MENZIES Max	Sydney Grammar School	NSW	10
MORRIS Anthony	Sydney Boys High School	NSW	11
BIUN John	Scotch College	VIC	10
ZHANG Alexander	James Ruse Agricultural High School	NSW	12

Silver

GARDAM Giles	Hurlstone Agricultural High School	NSW	10
BUDZYNOWSKI Aleksander	Sydney Grammar School	NSW	12
RANASINGHE Navin	Scotch College	VIC	12
THILLAINADESAN Senthil	Sydney Boys High School	NSW	12
SHAH Margit	James Ruse Agricultural High School	NSW	12
ZHU Jasmine	University High School	VIC	12
NORRISH Mark	University High School	VIC	11
SWEDOSH William	McKinnon Secondary College	VIC	11
HA Saleem Shing Kwan	James Ruse Agricultural High School	NSW	12
KEDZIORA David	Sydney Grammar School	NSW	12
YIN Albert	James Ruse Agricultural High School	NSW	12
WONG Kenneth	James Ruse Agricultural High School	NSW	12

LIANG Xin	James Ruse Agricultural High School	NSW	11
CHING Joshua	Cranbrook School	NSW	12
HOLLOWAY Jet	Hutchins School	TAS	12
WANG Morgan	Rossmoyne Senior High School	WA	12

Australian Team to participate in 2006 IMO at Ljubljana, Slovenia

Graham White, Charles Li, Matthew Ng, Vinoth Nandakumar, Konrad Pilch, Vinh Pham.

2007

Problems

PAPER 1 Tuesday 06 February

1. In the triangle ABC let AB be the shortest side. Let the midpoints of BC and AC be X and Y, respectively. Suppose P is on AC such that PX is perpendicular to BC. The circle passing through A, B and P meets the side BC again at Q.

Prove that QY is perpendicular to AC.

2. Let p be an odd prime.

Determine all pairs (m, n) of positive integers that satisfy

$$(p-1)(p^n+1) = 4m(m+1).$$

3. Consider an $m \times n$ rectangle divided into mn unit squares using $m-1$ horizontal grid lines and $n-1$ vertical grid lines. The rectangle is to be cut into mn unit squares in stages as follows.

The first stage is to cut the rectangle into two pieces along a grid line. Each subsequent stage consists of choosing one or more pieces and cutting each of these into two along one of its grid lines.

Find, in terms of m and n, the minimum number of stages required.

4. (a) Prove that for each positive integer n there exists a real number C_n such that

$$r + r^2 + r^3 + \cdots + r^{2n} \leq C_n(1 + r^{2n+1})$$

for all positive real numbers r.

(b) For each n, find the minimum value of C_n.

PAPER 2 Wednesday 07 February

5. Consider a 3×3 square array of distinct positive integers. It is called *special* if the eight products of the numbers in each row, each column and each diagonal are equal. For a special array let the common value of the products be P.

(a) Show that there is exactly one such P between 5000 and 6500.

(b) Give an example of a special array with this value of P.

6. Let a and b be non-zero real numbers such that $a^2 + b^2 = 1$.

Prove that $\left|a + \frac{a}{b} + b + \frac{b}{a}\right| \geq 2 - \sqrt{2}$.

7. Let ABC be an acute angled triangle. For each point M on the segment AC, let S_1 be the circle through A, B and M, and let S_2 be the circle through M, B and C. Let D_1 be the disk bounded by S_1, and let D_2 be the disk bounded by S_2.

Show that the area of the intersection of D_1 and D_2 is smallest when BM is perpendicular to AC.

8. Let S be a finite set of non-negative real numbers including 0. Suppose that whenever a and b belong to S, then at least one of $a + b$ and $|a - b|$ belongs to S.

Prove that

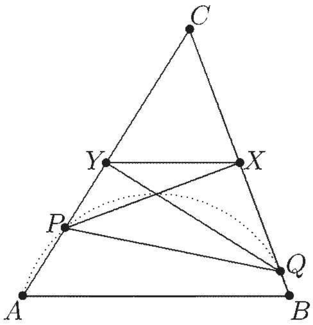
(i) S consists of exactly four distinct real numbers

or

(ii) $S = \{0, r, 2r, \ldots, nr\}$ for some non-negative integer n and some positive real number r.

Solution 1 (Anthony Morris, Sydney Boys High School, NSW)

See the diagram below. Since X and Y are the midpoints of BC and AC, this implies that $YX \parallel AB$. Thus $/CXY = /CBA$. We are given that $APQB$ is cyclic and hence $/QBA = /QPC$. Thus $/QPC = /CXY$. Hence $PQXY$ is a cyclic quadrilateral. Finally using this cyclic quadrilateral we see that $/QYP = /QXP = 90°$. Thus QY is perpendicular to AC.



Comment. From the diagram it seems that the circumcircle of triangle APB passes through the intersection point of PX and QY. It is in fact true and we leave this as an exercise for the reader.

Solution 2 (Kathryn Zealand, Brisbane Girls Grammar School, QLD)

The equation may be rewritten in the form

$$p(p^n - p^{n-1} + 1) = (2m + 1)^2.$$

If $n \geq 2$ then since the RHS is a perfect square and p is a prime we must have $p^2|\text{LHS}$. Thus $p|p^n - p^{n-1} + 1$. However $p|p^n$ and $p|p^{n-1}$, thus $p|1$ which is impossible. Thus there are no solutions when $n \geq 2$.

If $n = 1$, then $p = 2m + 1$. So $m = \frac{p-1}{2}$. Thus the set of solutions to the equation in positive integers m and n is $(m, n) = (\frac{p-1}{2}, 1)$.

Solution 3 (Max Menzies, Sydney Grammar School, NSW)

We claim the answer is $\lceil \log_2 m \rceil + \lceil \log_2 n \rceil$, where $\lceil x \rceil$ denotes the smallest integer not less than x. We prove this by strong induction on $k = m + n$.

If $k = 2$ then $m = n = 1$ and we require 0 cuts and hence 0 stages. If $k = 3$ then $m = 1$ and $n = 2$ or $m = 2$ and $n = 1$. Either way we need at least 1 cut and hence 1 stage. Note also that $\lceil \log_2 1 \rceil + \lceil \log_2 2 \rceil = 0 + 1 = 1$. Hence we have verified the base cases $k = 2, 3$.

Assume now that the claim is true for all positive integers m, n such that $m + n \leq k$. We now prove that it is also true for $m + n = k + 1$.

Consider an $m \times n$ rectangle with $m + n = k + 1$. After the first cut we will have cut this rectangle into two smaller rectangles. Suppose without loss of generality that we cut along one of the $n - 1$ vertical grid lines. This yields two rectangles of dimensions $m \times a$ and $m \times b$ respectively where $a + b = n$. Suppose without loss of generality that $a \geq b$, then also $a \geq \frac{n}{2}$. The $m \times a$ rectangle satisfies $m + a \leq k$ and so by our inductive assumption we need at least a further $\lceil \log_2 m \rceil + \lceil \log_2 a \rceil$ stages for this rectangle alone. Thus the number of stages for the original $m \times n$ rectangle is at least

$$\begin{aligned}
1 + \lceil \log_2 m \rceil + \lceil \log_2 a \rceil &= \lceil \log_2 m \rceil + \lceil \log_2 a + 1 \rceil \\
&= \lceil \log_2 m \rceil + \lceil \log_2 2a \rceil \\
&\geq \lceil \log_2 m \rceil + \lceil \log_2 n \rceil \text{ (since } a \geq \tfrac{n}{2}\text{)}.
\end{aligned}$$

Thus we have proven by strong induction that we need at least $\lceil \log_2 m \rceil + \lceil \log_2 n \rceil$ stages to reduce our $m \times n$ rectangle to mn unit squares.

To complete the proof we provide a method that yields mn unit squares after exactly $\lceil \log_2 m \rceil + \lceil \log_2 n \rceil$ stages. Suppose that $2^{k-1} < m \leq 2^k$ and $2^{l-1} < n \leq 2^l$. Let us augment the $m \times n$ rectangle to a $2^k \times 2^l$ rectangle. It suffices to show that for this $2^k \times 2^l$ rectangle we can cut it into unit squares using just $k + l$ stages. This is because if we can reduce the $2^k \times 2^l$ rectangle to unit squares in $k + l$ stages then the restriction of the same cuts at each stage to the $m \times n$ rectangle also reduces it to unit squares at the same time. However the $2^k \times 2^l$ rectangle is easy to deal with. We simply cut all the pieces in half horizontally at each stage to arrive at 2^k identical horizontal strips of dimension 1×2^l. This requires k stages. Then we cut all the pieces in half vertically at each stage to arrive at 2^{k+l} unit squares after a further l stages. Thus the total number of stages for this method is $k + l = \lceil \log_2 m \rceil + \lceil \log_2 n \rceil$. This completes the proof.

Solution 4

Alternative 1 (There were a number of solutions that took the same path for the initial part of the solution as shown below. After this we present three variants which complete the solution.)

Let $r = 1$. This immediately yields $C_n \geq n$. We prove that $C_n = n$. If we could prove the following inequalities

$$\begin{aligned}
r + r^{2n} &\leq 1 + r^{2n+1} \\
r^2 + r^{2n-1} &\leq 1 + r^{2n+1} \\
&\vdots \\
r^n + r^{n+1} &\leq 1 + r^{2n+1}
\end{aligned}$$

then upon adding these n inequalities together we would in fact have shown

$$r + r^2 + r^3 + \cdots + r^{2n} \leq n(1 + r^{2n+1})$$

which would complete the problem. Thus it suffices to prove

$$r^i + r^{2n+1-i} \leq 1 + r^{2n+1}$$

for any integer i such that $1 \leq i \leq n$. We now present the variants which do this.

Alternative 1, Variant 1 (Vinoth Nandakumar, James Ruse Agricultural High School, NSW)

Note that $r^{2n+1} + 1 - r^i - r^{2n+1-i} = (r^i - 1)(r^{2n+1-i} - 1)$. If $r < 1$, both brackets are negative. If $r = 1$, both brackets are zero. If $r > 1$, both brackets are positive. Hence in all cases the inequality is proven.

Alternative 1, Variant 2 (Giles Gardam, Hurlstone Agricultural High School, NSW)

Whether or not $r \geq 1$ or $r \leq 1$, the sets $\{1, r^i\}$ and $\{1, r^{2n+1-i}\}$ are similarly ordered (i.e. both are increasing or decreasing). Thus by the Rearrangement Inequality we know that $1 \cdot 1 + r^i \cdot r^{2n+1-i} \geq 1 \cdot r^{2n+1-i} + r^i \cdot 1$. This is the desired inequality.

Alternative 1, Variant 3 (Irene Lo, James Ruse Agricultural High School, NSW)

By Muirhead's inequality $x^{2n+1} + y^{2n+1} \geq x^i y^{2n+1-i} + x^{2n+1-i} y^i$ for non-negative real numbers x, y, since the vector $(2n+1, 0)$ majorizes the vector $(2n + 1 - i, i)$. The result follows if we put $x = r$ and $y = 1$.

Alternative 2 (Andrew Elvey Price, Brunswick Secondary College, VIC)

By the AM-GM Inequality we have for any integer i such that $1 \leq i \leq 2n$

$$\frac{(2n + 1 - i) \cdot 1 + i \cdot r^{2n+1}}{2n + 1} \geq r^i.$$

Let us add these results together for $i = 1$ to $2n$. This yields

$$n(1 + r^{2n+1}) \geq r + r^2 + r^3 + \cdots + r^{2n}.$$

Therefore C_n can equal n. Note that when $r = 1$ we have equality when $C_n = n$, then n is the minimum possible value of C_n.

Alternative 3 (Ildar Gaisin, All Saints Anglican School, QLD)

Putting $r = 1$ shows that $C_n \geq n$. We prove $C_n = n$. Note that the LHS factors as $(r + 1)(r + r^3 + r^5 + \cdots + r^{2n-1})$ and the RHS factors as $n(1 + r)(r^{2n} - r^{2n-1} + r^{2n-2} - \cdots + r^2 - r + 1)$. Since $r > 0$ it suffices to prove that

$$r + r^3 + r^5 + \cdots + r^{2n-1} \leq n(r^{2n} - r^{2n-1} + r^{2n-2} - \cdots + r^2 - r + 1).$$

This is equivalent to showing $P(r) \geq 0$ where

$$P(r) = nr^{2n} - (n+1)r^{2n-1} + nr^{2n-2} - (n+1)r^{2n-3} + \cdots + nr^2 - (n+1)r + n.$$

If it could be shown that $P(r)$ factorizes as follows then this would prove that $P(r) \geq 0$ since it is a product of a perfect square and a polynomial with positive coefficients.

$$P(r) = (r-1)^2 \sum_{i=2}^{2n} \left\lfloor \frac{i}{2} \right\rfloor \left(n - \left\lfloor \frac{i-1}{2} \right\rfloor \right) r^{2n-i}$$

Expanding the RHS of the above factorisation, we see that the coefficient of r^{2n-j} $(0 \leq j \leq 2n)$ is

$$f(j) = \left\lfloor \tfrac{j}{2} \right\rfloor \left(n - \left\lfloor \tfrac{j-1}{2} \right\rfloor\right) - 2\left\lfloor \tfrac{j+1}{2} \right\rfloor \left(n - \left\lfloor \tfrac{j}{2} \right\rfloor\right) + \left\lfloor \tfrac{j+2}{2} \right\rfloor \left(n - \left\lfloor \tfrac{j+1}{2} \right\rfloor\right).$$

If j is even then $j = 2j_1$ for some integer j_1 with $0 \leq j_1 \leq n$, and

$$f(j) = f(2j_1) = j_1(n - j_1 + 1) - 2j_1(n - j_1) + (j_1 + 1)(n - j_1) = n.$$

If j is odd then $j = 2j_1 + 1$ for some integer j_1 with $0 \leq j_1 \leq n-1$, and

$$\begin{aligned} f(j) &= f(2j_1 + 1) \\ &= j_1(n - j_1) - 2(j_1 + 1)(n - j_1) + (j_1 + 1)(n - j_1 - 1) \\ &= -n - 1. \end{aligned}$$

Thus we have proven that the factorisation is correct and this concludes the solution.

Solution 5

Alternative 1 (Sampson Wong, James Ruse Agricultural High School, NSW; Bonnie Zhang, James Ruse Agricultural High School, NSW)

Let us label some the squares of the 3×3 array as follows

a	c	
	b	

Since the product along each column row and diagonal equals P we may write $\frac{P}{ac}$ in the top right square, then $\frac{ac}{b}$ in the bottom left square, $\frac{P}{ab}$ in the bottom right square. Finally considering the centre column of squares we may write $\frac{P}{bc}$ in the bottom middle square. This gives us

a	c	$\frac{P}{ac}$
	b	
$\frac{ac}{b}$	$\frac{P}{bc}$	$\frac{P}{ab}$

Now the product of the numbers in the bottom row equals P, so $P = \frac{ac}{b} \cdot \frac{P}{bc} \cdot \frac{P}{ab}$ which simplifies to $P = b^3$. Hence P is a perfect cube lying between 5000 and 6500. Since $17^3 = 4913$, $18^3 = 5832$ and $19^3 = 6859$ we see that only $P = 5832$ is permissible.

Finally an example of a special array with this value of P is

6	27	36
108	18	3
9	12	54

Alternative 2 (Giles Gardam, Hurlstone Agricultural High School, NSW)

Let us label the special array as shown.

a	b	c
d	e	f
g	h	i

By the special property we know $P = abc = def = ghi$, so $P^3 = abcdefghi$. Also by the special property we know $P = aei = beh = ceg = fed$, so $P^4 = abcde^4fghi$. If we divide the expression for P^4 by the expression for P^3 we obtain $P = e^3$. Thus P is a perfect cube with $5000 \leq P \leq 6500$. Since $17^3 = 4913$, $18^3 = 5832$ and $19^3 = 6859$ we see that only $P = 5832$ is permissible.

An example of such a special array is

27	4	54
36	18	9
6	81	12

Solution 6 (Ildar Gaisin, All Saints Anglican School, QLD; Andrew Elvey Price, Brunswick Secondary College, VIC)

By the triangle inequality we have

$$\left| a + \frac{a}{b} + b + \frac{b}{a} \right| \geq \left| \frac{a}{b} + \frac{b}{a} \right| - |a + b|.$$

Since $\frac{a}{b}$ and $\frac{b}{a}$ have the same sign we know that $\left|\frac{a}{b} + \frac{b}{a}\right| = \frac{|a|}{|b|} + \frac{|b|}{|a|}$. Furthermore we may now apply the AM-GM (Arithmetic Mean – Geometric Mean) Inequality to derive

$$\left| \frac{a}{b} + \frac{b}{a} \right| \geq 2.$$

Thus it suffices to show that $|a+b| \leq \sqrt{2}$. However this follows easily from the triangle inequality and the QM-AM (Quadratic Mean – Arithmetic Mean) Inequality since

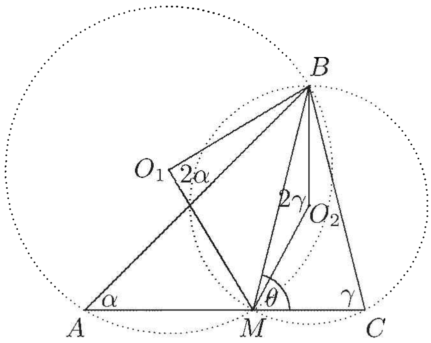
$$\begin{aligned} \frac{|a+b|}{2} &\leq \frac{|a|+|b|}{2} \\ &\leq \sqrt{\frac{|a|^2+|b|^2}{2}} \\ &= \frac{1}{\sqrt{2}}. \end{aligned}$$

Comment: We can determine when equality occurs. In the AM-GM and the QM-AM, equality could only occur when $|a| = |b|$. In the triangle inequality $|a+b| \leq |a| + |b|$, equality can only occur if a and b have the same sign. Thus a necessary condition for equality is $a = b$. Since $a^2 + b^2 = 1$, this means that $(a,b) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ or $(a,b) = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. When we check these values by substitution these we see that only the second pair yields equality.

Solution 7

Alternative 1 (Ildar Gaisin, All Saints Anglican School, QLD)

Let $\angle BAC = \alpha$, $\angle BCA = \gamma$, $BC = a$ and $AB = c$. Let O_1 and O_2 denote the circumcentres of triangles AMB and BMC, respectively. Let $\angle BMC = \theta$. Let A_1 denote the area contained in the minor segment bounded by BM in D_1 and let A_2 denote the area contained in the minor segment bounded by BM in D_2. Refer to the figure below.



We want to minimize $A_1 + A_2$. We know that $\angle BO_1M = 2\alpha$. Let R_1 and R_2 denote the radii of S_1 and S_2 respectively. The area of the sector bounded by BO_1M equals $\frac{2\alpha}{360^\circ}\pi R_1^2$. The area of triangle BO_1M

is $\frac{1}{2}R_1^2\sin 2\alpha$. Hence

$$\begin{aligned} A_1 &= R_1^2\left(\tfrac{\alpha\pi}{180^\circ}-\tfrac{1}{2}\sin 2\alpha\right), \quad \text{and similarly} \\ A_2 &= R_2^2\left(\tfrac{\gamma\pi}{180^\circ}-\tfrac{1}{2}\sin 2\gamma\right). \end{aligned}$$

By the extended sine rule in triangles BMC and AMB we have $2R_2 = \frac{a}{\sin\theta}$ and $2R_1 = \frac{c}{\sin(180^\circ-\theta)} = \frac{c}{\sin\theta}$. We now compute

$$\begin{aligned} A_1+A_2 &= \tfrac{c^2}{4\sin^2\theta}\left(\tfrac{\alpha\pi}{180^\circ}-\tfrac{1}{2}\sin 2\alpha\right)+\tfrac{a^2}{4\sin^2\theta}\left(\tfrac{\gamma\pi}{180^\circ}-\tfrac{1}{2}\sin 2\gamma\right) \\ &= \tfrac{K}{\sin^2\theta} \end{aligned}$$

where $K = \frac{1}{4}(\frac{\alpha\pi c^2}{180^\circ}+\frac{\gamma\pi a^2}{180^\circ}-\frac{1}{2}c^2\sin 2\alpha-\frac{1}{2}a^2\sin 2\gamma)$. Note that K is a constant since it depends only on a, c, α, γ which are all fixed in triangle ABC. Thus the minimum of A_1+A_2 occurs where $\sin\theta$ is maximized. This occurs when $\theta = 90^\circ$ and corresponds to $BM \perp AC$.

Alternative 2 (Sampson Wong, James Ruse Agricultural High School, NSW)

Using the notation as in the first alternative we may compute $A_1 = k_1R_1^2$ and $A_2 = k_2R_2^2$ where k_1 and k_2 are constants depending only on triangle ABC. By the triangle inequality in triangle AO_1B we have $AO_1+BO_1 \geq AB$. Hence $R_1 \geq \frac{1}{2}AB$. Similarly $R_2 \geq \frac{1}{2}BC$. Hence $A_1+A_2 \geq \frac{1}{4}k_1AB^2+\frac{1}{4}k_2BC^2$ which is a constant since it depends only on triangle ABC. Equality occurs if and only if AO_1B and CO_1B are straight lines. This is possible precisely when AB and BC are diameters of S_1 and S_2. This in turn occurs precisely when BM is perpendicular to AC.

Solution 8 (Vinoth Nandakumar, James Ruse Agricultural High School, NSW; Ildar Gaisin, All Saints Anglican School, QLD)

If S consists of exactly 1 or 2 elements then $S = \{0\}$ or $S = \{0, p\}$, where $p > 0$. Both these cases clearly satisfy condition (ii). If S consists of exactly 3 elements say $S = \{0, p, q\}$ where $0 < p < q$, then since $q+p > q$, we have $q-p \in S$. Since $0 < q-p < q$ this means $q-p = p$ so that $S = \{0, p, 2p\}$, which clearly satisfies (ii). If S consists of exactly 4 elements then S satisfies condition (i), but we will see what S looks like anyway. Suppose $S = \{0, p, q, r\}$ where $0 < p < q < r$, then since $r+q, r+p > r$ we have $r-q, r-p \in S$. Since $0 < r-q < r-p < r$ then these four elements must be exactly $0, p, q, r$ in that order. This means that $r = p+q$ so that $S = \{0, p, q, p+q\}$ which is clearly seen to be a valid set for S.

Suppose now that S has at least 5 elements. Write $S = \{0, x_1, x_2, \ldots, x_n\}$ where $0 < x_1 < x_2 < \ldots < x_n$. Consider the pair (x_n, x_i) for any i such that $1 \leq i \leq n-1$. If $x_n + x_i \in S$ then we contradict the choice that x_n is the greatest element of S. Thus $x_n - x_i \in S$ for $i = 1, 2, \ldots, n-1$. However the following two lists

$$\begin{array}{ccccccccccc} 0 & < & x_n - x_{n-1} & < & x_n - x_{n-2} & < & \ldots & < & x_n - x_1 & < & x_n \\ 0 & < & x_1 & < & x_2 & < & \ldots & < & x_{n-1} & < & x_n \end{array}$$

both represent all of the elements of S in increasing order. Hence they must be equal term for term. Hence

$$x_n = x_{n-i} + x_i \text{ for } i = 1, 2, \ldots, n-1. \ (*)$$

Consider the pair (x_{n-1}, x_i) for any i such that $2 \leq i \leq n-2$. If $x_{n-1} + x_i \in S$ then from (*) we have $x_n = x_{n-1} + x_1 < x_{n-1} + x_i$ (since $i \geq 2$). This contradicts that x_n was chosen to be the greatest element of S. Thus $x_{n-1} - x_i \in S$ for $i = 2, 3, \ldots, n-2$. Let us compare the following two lists of elements of S listed in increasing order

$$\begin{array}{ccccccccccc} 0 & < & x_{n-1} - x_{n-2} & < & \ldots & < & x_{n-1} - x_2 & < & x_{n-1} & < & x_n \\ 0 & < & x_1 & < & \ldots & < & x_{n-2} & < & x_{n-1} & < & x_n. \end{array}$$

Note that the first list has n members of S whereas the second list has all $n+1$ members of S. Thus $x_{n-1} - x_2$ is either the third largest or fourth largest element of S. If it is the third largest then $x_{n-1} - x_2 = x_{n-2}$. Thus $x_{n-1} = x_2 + x_{n-2} = x_n$ from (*), a contradiction. Thus $x_{n-1} - x_2$ is the fourth largest element of S. Hence the following two lists of numbers

$$\begin{array}{ccccccccc} 0 & < & x_{n-1} - x_{n-2} & < & x_{n-1} - x_{n-3} & < & \ldots & < & x_{n-1} - x_2 \\ 0 & < & x_1 & < & x_2 & < & \ldots & < & x_{n-3} \end{array}$$

both represent the first $n-2$ elements of S and so they are equal term for term. This means that

$$x_{n-1} = x_{n-i} + x_{i-1} \text{ for } i = 3, 4, \ldots, n-1.$$

Subtracting this from (*) we find

$$x_i - x_{i-1} = x_n - x_{n-1} \text{ for } i = 3, 4, \ldots, n-1.$$

Since $x_n - x_{n-1} = x_1$ from (*), this means that $x_2, x_3, \ldots, x_n$ forms an arithmetic sequence with common difference x_1. Thus we may write $x_i = x_2 + (i-2)x_1$ for $i = 2, 3, \ldots, n$. Now consider the pair (x_n, x_{n-2}). Their sum cannot be in S since this is larger than x_n. Hence their difference must be in S. Since S has at least 5 elements, this means that $n \geq 4$ and so $n - 2 \geq 2$. Thus $x_n - x_{n-2} = 2x_1$ is in S. Since $x_1 < 2x_1 < x_2 + x_1 = x_3$ we have that $x_2 = 2x_1$. This implies that $0, x_1, x_2, \ldots, x_n$ is an arithmetic sequence with common difference x_1 and hence satisfies condition (ii).

RESULTS

Number of Candidates: 99

Gold

GAISIN Ildar	All Saints Anglican School	QLD	12
NANDAKUMAR Vinoth	James Ruse Agricultural High School	NSW	12
MENZIES Max	Sydney Grammar School	NSW	11
ELVEY PRICE Andrew	Brunswick Secondary College	VIC	10
LO Irene	James Ruse Agricultural High School	NSW	11
MORRIS Anthony	Sydney Boys High School	NSW	12
ZHANG Bonnie	James Ruse Agricultural High School	NSW	11
BIUN John	Scotch College	VIC	11
LIN Sen	James Ruse Agricultural High School	NSW	11
WONG Sampson	James Ruse Agricultural High School	NSW	9

Silver

LUO May	St George Girls High School	NSW	12
CHEUNG Paul	Sydney Technical High School	NSW	11
GARDAM Giles	Hurlstone Agricultural High School	NSW	11
GARDINER Sean	Sydney Grammar School	NSW	11
STRUGNELL Sebastian	Trinity Grammar School	VIC	12
ZEALAND Kathryn	Brisbane Girls Grammar School	QLD	11
CAMERON Harry	Sydney Grammar School	NSW	11
LIANG Xin	James Ruse Agricultural High School	NSW	12
REITER Perry	Scotch College	VIC	11
ZHU Jonathan	Smiths Hill High School	NSW	10
GIRDHAR Parth	James Ruse Agricultural High School	NSW	10

ZHOU Peter	Ivanhoe Grammar School	VIC	11
CHONG Aaron	Doncaster Secondary College	VIC	9
LUK Jeremy	James Ruse Agricultural High School	NSW	11
ZHOU Qiming	James Ruse Agricultural High School	NSW	12

Australian Team to participate in 2007 IMO at Hanoi, Vietnam

Ildar Gaisin, Max Menzies, Irene Lo, Anthony Morris, Sen Lin, Giles Gardam.

2008

Problems

PAPER 1 Tuesday 05 February

1. Let K be a circle with PQ as a diameter. Let C be a circle with centre on K and with PQ tangent to C.

Prove that the other tangents from P and Q are parallel.

2. Let $f(x) = 5^x$.

Determine all real solutions of the equation

$$f(x + f(2008)) = 2008 - x.$$

3. A positive integer is called *square-free* if it has no factor greater than 1 which is a perfect square.

For each positive integer n, let $f(n)$ be the sum of all square-free factors of n.

Determine all values of n for which $\dfrac{f(n)}{n}$ is an integer.

4. Find all positive integers n and all prime numbers p such that the polynomial

$$x^5 + x + p^n$$

can be written as the product of two polynomials with integer coefficients and positive degrees.

PAPER 2 Wednesday 06 February

5. For each positive integer m, let $F(m)$ be the largest integer such that $10^{F(m)}$ divides $m!$, where $m! = 1 \times 2 \times \cdots \times (m-1) \times m$.

Prove that there exists a positive integer n such that, for each m,

$$\textbf{either} \quad F(m) \leq n \quad \textbf{or} \quad F(m) \geq n + 2008.$$

6. Let $ABCD$ be a convex quadrilateral. Suppose there is a point P on the segment AB with $\angle APD = \angle BPC = 45^\circ$.

If Q is the intersection of the line AB with the perpendicular bisector of CD, prove that $\angle CQD = 90^\circ$.

7. Let $A_1A_2A_3$ and $B_1B_2B_3$ be triangles.

If

$$p = A_1A_2 + A_2A_3 + A_3A_1 + B_1B_2 + B_2B_3 + B_3B_1$$

and

$$\begin{aligned} q \; = \; & A_1B_1 + A_1B_2 + A_1B_3 + A_2B_1 + A_2B_2 + A_2B_3 \\ & + A_3B_1 + A_3B_2 + A_3B_3 \end{aligned}$$

prove that

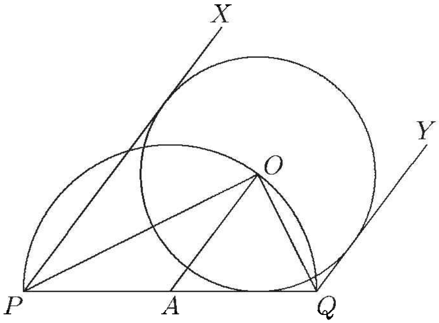
$$3p \leq 4q.$$

8. A rectangular chessboard has 5 rows and 2008 columns. Each square is painted either red or blue.

Determine the largest integer N which guarantees that, no matter how the board is painted, there are two rows which have matching colours in at least N columns.

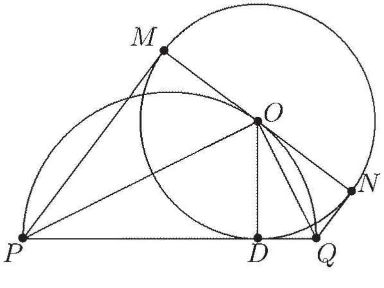
Solution 1

Alternative 1 (Jialing Chin, Alstonville High School, NSW)



Let X and Y be points such that PX and QY are tangents to circle C. Let A and O be the centres of circles K and C, respectively. Thus $AO = AP$. Hence $\angle AOP = \angle OPA$. Since PX and PQ are both tangents meeting at P, it follows that PO bisects $\angle XPQ$. Therefore $\angle OPX = \angle OPA = \angle AOP$. Hence $PX \parallel AO$. Similarly $QY \parallel AO$. Thus PX is parallel to QY.

Alternative 2 (Aaron Chong, Doncaster Secondary College, VIC)



Let the points of contact of the other tangents from P and Q to C be M and N, respectively. Let C be tangent to PQ at point D. Since a tangent is perpendicular to the radius, it follows that $OM \perp PM$, $OD \perp PQ$ and $ON \perp QN$. So, it suffices to prove that points M, O and N are collinear.

Triangles QDO and QNO are congruent because $DO = NO$, $QO = QO$ and $\angle ODQ = \angle ONQ$. Thus $\angle QOD = \angle QON$. Similarly $\angle POD = \angle POM$. Hence $\angle MON = \angle POM + \angle POD + \angle QOD + \angle QON = 2\angle POD + 2\angle QOD = 2\angle POQ = 180^\circ$. Thus M, O and N are collinear, as required.

Solution 2 (Jonathan Zhu, Smith's Hill High School, NSW)

Let $y = x + 5^{2008}$. This simplifies the equation to

$$5^y + y = 5^{2008} + 2008.$$

Note that $y = 2008$ solves the above equation. Since the LHS of the above equation is a strictly increasing function and the RHS is constant, there is at most one value of y satisfying the equation. Thus there are no further solutions. Since $x = y - 5^{2008}$ we see that $x = 2008 - 5^{2008}$ is the only solution to the original equation.

Solution 3 (Max Menzies, Sydney Grammar School, NSW)

Note that $n = 1$ is a solution.

If $n > 1$ then we may write n in its prime factorisation

$$n = {p_1}^{\alpha_1}{p_2}^{\alpha_2}\dots{p_k}^{\alpha_k},$$

where $k \geq 1$, $\alpha_1, \alpha_2, \dots, \alpha_k \geq 1$ and $p_1 < p_2 < \dots < p_k$.

The square-free divisors of n are

$$\begin{gathered} 1, \\ p_1,\ p_2,\ \dots,\ p_k, \\ p_1p_2,\ p_1p_3,\ \dots,\ p_{k-1}p_k, \\ p_1p_2p_3,\ p_1p_2p_4,\ \dots,\ p_{k-2}p_{k-1}p_k, \\ \vdots \\ p_1p_2\cdots p_k. \end{gathered}$$

We note that each of these terms appears exactly once in the expansion of $(1+p_1)(1+p_2)\dots(1+p_k)$. Thus

$$f(n) = (1+p_1)(1+p_2)\dots(1+p_k).$$

For $n|f(n)$ we certainly require that

$$p_k|(1+p_1)(1+p_2)\dots(1+p_k).$$

Since p_k is prime it follows that $p_k|(1+p_i)$ for some $i \in \{1, 2, \dots, k\}$. But $p_k \nmid 1 + p_k$, thus $k \geq 2$ and $i \leq k-1$.

Note that $p_k > p_{k-1} > p_{k-2} > \dots$ quickly implies that $p_k > p_j + 1$ for any $j \leq k-2$. So $p_k \nmid 1 + p_j$ for any $j \leq k-2$. Thus we must have $p_k|1+p_{k-1}$. So $p_k \leq 1+p_{k-1}$. But $p_k > p_{k-1}$. Thus forces $p_k = 1+p_{k-1}$. The only two primes which are consecutive integers are 2 and 3. Thus $p_k = 3$ and $p_{k-1} = 2$. So 2 and 3 must be the two largest primes dividing n. This implies that they are the only primes dividing n.

From here we see that $f(n) = 1 + 2 + 3 + 6 = 12$. Since $n|f(n)$, and n is divisible by 2 and 3, it follows that $n = 6$ or $n = 12$. We check that $\frac{f(6)}{6} = 2$ and $\frac{f(12)}{12} = 1$ are both integers. Thus the solutions are $n = 1$, 6 or 12.

Solution 4

Alternative 1 (Giles Gardam, Hurlstone Agricultural High School, NSW)

There are two cases we must consider. Either the polynomial may be written as the product of a linear polynomial and a quartic polynomial or it may be written as the product of a quadratic and a cubic polynomial. In either case since we are dealing with integer polynomials, and since x^5+x+p^n is monic, we may assume that the two factors are also monic.

Case 1. Linear and quartic factors. We may assume that the linear factor has the form $x - a$ for some integer a. Thus $x = a$ is a root of $x^5 + x + p^n$. Hence

$$\begin{array}{rrcl} & a^5 + a + p^n & = & 0 \\ \Rightarrow & a(a^4 + 1) & = & -p^n. \end{array}$$

If $|a| \geq 2$ then a is divisible by some prime, q say. Since $\gcd(a, a^4+1) = 1$ and $a^4 + 1 \geq 2^4 + 1 > 1$, it follows that $a^4 + 1$ must be divisible by a different prime other than q. Thus two different primes would be factors of p^n, which is impossible. Thus $a = -1, 0, 1$. These in turn yield $p^n = 2, 0, -2$. Only $p^n = 2$ is feasible for which $p = 2$ and $n = 1$. Note that for this case we have $x^5 + x + 2 = (x + 1)(x^4 - x^3 + x^2 - x + 2)$. So $(n, p) = (1, 2)$ is a solution.

Case 2. Quadratic and cubic factors. Assume that we may write the factorization as

$$x^5 + x + p^n = (x^3 + ax^2 + bx + c)(x^2 + dx + e).$$

Upon expansion and equating of coefficients we end up with the following five equations.

$$\begin{array}{rcl} d + a & = & 0 \\ e + ad + b & = & 0 \\ ae + bd + c & = & 0 \\ be + cd & = & 1 \\ ce & = & p^n. \end{array}$$

We substitute $d = -a$ into the other four equations and find

$$\begin{array}{rcl} b + e - a^2 & = & 0 \\ a(e - b) + c & = & 0 \\ be - ac & = & 1 \\ ce & = & p^n. \end{array}$$

Consider if both $p|c$ and $p|e$. Then $p|be-ac$, so that $p|1$, which is impossible. Thus c and e are two integers with product p^n and at most one of them is divisible by p. This leads us to four cases for the values of c and e. We will show that each case leads to a contradiction.

Case 2a. $c=p^n$, $e=1$. Substitution into the remaining three equations yields

$$\begin{aligned} b+1-a^2 &= 0 \qquad (1)\\ a(1-b)+p^n &= 0 \qquad (2)\\ b-ap^n &= 1. \qquad (3) \end{aligned}$$

Equation (3) tells us $b=1+ap^n$. If we substitute this into (2) we find $p^n(1-a^2)=0$. Since $p^n\neq 0$ it must be that $a^2=1$. However from (1) this implies that $b=0$. Going back to (3) we find $ap^n=-1$ which is impossible since $p\geq 2$ and $n\geq 1$.

Case 2b. $c=-p^n$, $e=-1$. Substitution into the remaining three equations yields

$$\begin{aligned} b-1-a^2 &= 0 \qquad (1)\\ a(1+b)+p^n &= 0 \qquad (2)\\ ap^n-b &= 1. \qquad (3) \end{aligned}$$

Equation (3) tells us $b=ap^n-1$. If we substitute this into (2) we find $p^n(1+a^2)=0$ which is impossible since $p>0$ and $a^2+1>0$.

Case 2c. $c=1$, $e=p^n$. Substitution into the remaining three equations yields

$$\begin{aligned} b+p^n-a^2 &= 0 \qquad (1)\\ a(p^n-b)+1 &= 0 \qquad (2)\\ p^nb-a &= 1. \qquad (3) \end{aligned}$$

Equation (2) tells us that $a|1$ and $p^n-b|1$. Thus $a=\pm 1$ and $b-p^n=\pm 1$. Either way $a^2=1$. Substituting this into (1) yields $b=1-p^n$. Substituting this value of b into $b-p^n=\pm 1$ yields $1-2p^n=\pm 1$. Thus $p^n=0$ or $p^n=1$. Both options are impossible since $p>1$ and $n>0$.

Case 2d. $c=-1$, $e=-p^n$. Substitution into the remaining three equations yields

$$\begin{aligned} b-p^n-a^2 &= 0 \qquad (1)\\ a(p^n+b)+1 &= 0 \qquad (2)\\ a-p^nb &= 1. \qquad (3) \end{aligned}$$

Again (2) tells us that $a|1$ and $b+p^n|1$. Thus $a=\pm 1$ and $b+p^n=\pm 1$. Substituting $a^2=1$ into (1) yields $b=p^n+1$. Substituting this value of b into $b+p^n=\pm 1$ yields $2p^n+1=\pm 1$. Thus $p^n=0$ or $p^n=-1$, both of which are impossible.

Thus we have shown that the second case leads to no further solutions.

Alternative 2 (Norman Do)

Assume that we have a factorisation

$$x^5 + x + p^n = (a + bx + \cdots)(c + dx + \cdots) = A(x)B(x)$$

into monic polynomials. (N.B. If one of these factors is linear then $b = 1$ or $d = 1$.) By comparing the coefficients of the constant terms and then also the linear terms we find

$$\begin{aligned} ac &= p^n \\ bc + ad &= 1. \end{aligned}$$

If both $p|a$ and $p|c$, then we would also have $p|1$ from the second equation. Thus a, c are not both divisible by p. Thus one of a, c equals ± 1. Without loss of generality say $a = \pm 1$.

Consider all complex roots $z_1, z_2, \ldots, z_n$ of $A(x) = a + bx + \cdots$. By Viete's formulae $\prod_{i=1}^{n} |z_i| = |a| = 1$. This is impossible if $|z_i| > 1$ for $i = 1, 2, \ldots, n$. Thus $A(x)$ has a root z such that $|z| \leq 1$. But z is also a root of the original polynomial. Hence

$$\begin{aligned} |p^n| &= |-z - z^5| \\ &\leq |z| + |z|^5 \\ &\leq 2. \end{aligned}$$

Hence $p^n = 2$. This yields $p = 2$ and $n = 1$ for which we do have the factorization $x^5 + x + 2 = (x + 1)(x^4 - x^3 + x^2 - x + 2)$.

Solution 5

Alternative 1 (Sampson Wong, James Ruse Agricultural High School, NSW)

We claim that $n = F(10^{2008} - 1)$ has the desired property.

If $m \leq 10^{2008} - 1$, then $m!|(10^{2008} - 1)!$, so $m!$ cannot have more than $F(10^{2008} - 1)$ factors of 10. Thus in this case $F(m) \leq n$.

If $m \geq 10^{2008}$, then

$$\begin{aligned} & 10^{2008} \times (10^{2008} - 1)!|m! \\ \Rightarrow\quad & 10^{2008} \times 10^{F(10^{2008}-1)}|m! \\ \Leftrightarrow\quad & 10^{2008+n}|m!. \end{aligned}$$

Thus in this case $F(m) \geq n + 2008$.

Alternative 2 (Aaron Chong, Doncaster Secondary College, VIC)

We claim that $n = F(5^{2008} - 1)$ has the desired property.

We first note that $F(m)$ is non-decreasing. This is because $10^{F(m)}|m!$ implies that $10^{F(m)}|(m + 1)!$ so that $F(m) \leq F(m + 1)$.

The highest power of 5 dividing $m!$ can be determined as follows: For each multiple of 5 between 1 and m inclusive, add one to the count. For each multiple of 25, add an extra one to the count, as each contributes two 5s, one of which has already been counted. For each multiple of 125, add one to the count, and so on. Thus the highest power of five that divides $m!$ is $\lfloor \frac{m}{5} \rfloor + \lfloor \frac{m}{5^2} \rfloor + \lfloor \frac{m}{5^3} \rfloor + \cdots$. Similarly the highest power of 2 that divides $m!$ is $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2^2} \rfloor + \lfloor \frac{m}{2^3} \rfloor + \cdots$ which is term for term greater than the highest power of 5 that divides $m!$. Thus the highest power of 10 that divides $m!$ equals the highest power of 5 that divides $m!$.

Let $M = 5^{2008}$. Note that $F(M) = 5^{2007} + 5^{2006} + \cdots + 5 + 1$ and that

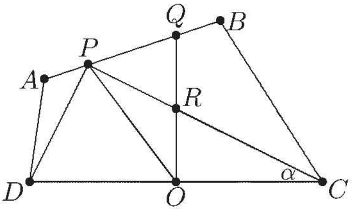
$$\begin{aligned} F(M-1) &= (5^{2007}-1) + (5^{2006}-1) + \cdots + (5-1) + (1-1) \\ &= F(M) - 2008. \end{aligned}$$

Let $n = F(M) - 2008$. Since $F(m)$ is non-decreasing it follows that if $m \leq M - 1$ then $F(m) \leq F(M-1) = n$ and that if $m \geq M$ then $F(m) \geq F(M) = n + 2008$. Thus such an n exists. In fact we may explicitly calculate the value considered in this proof to be

$$n = F(5^{2008} - 1) = \frac{5^{2008} - 1}{4} - 2008.$$

Solution 6

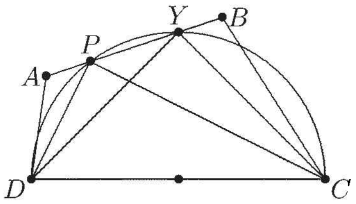
Alternative 1 (Joule Li, St Peter's College, SA)



Let O be the midpoint of CD, let QO intersect CP at R and let $\angle PCD = \alpha$. Note that $\angle DPC = 180° - \angle APD - \angle BPC = 90°$. Thus O is the centre of the circumcircle of triangle DPC. Thus $OD = OP = OC$. Thus triangle POC is isosceles and so $\angle OPC = \angle OCP = \alpha$. Thus $\angle OPQ = 45° + \alpha$. From the angle sum in triangle ROC we see that $\angle ORC = 90° - \alpha$. Hence $\angle PRQ = 90° - \alpha$.

Finally from the angle sum in triangle PQR we see that $\angle PQR = 180° - \angle QPR - \angle QRP = 180° - 45° - (90° - \alpha) = 45° + \alpha$. Thus $\angle OPQ = \angle OQP$ which implies that $OP = OQ$. Hence Q lies on the circle centred at O, radius OP. Since CD is a diameter of this circle, it follows that $\angle CQD = 90°$.

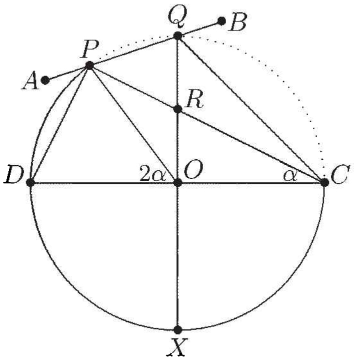
Alternative 2 (Sampson Wong, James Ruse Agricultural High School, NSW, using the technique of "Reverse Reconstruction")



Let Γ be the circle with diameter CD. Let Y be the second point of intersection of Γ with line AB. Since $PYCD$ is cyclic it follows that $/YDC = /YPC = 45^\circ$. Also $/YCD = /APD = 45^\circ$ from the exterior angle of a cyclic quadrilateral. Thus triangle YDC is isosceles with $YD = YC$. Thus Y lies on the perpendicular bisector of CD. However since Y also lies on the line AB, it follows that Y is the intersection of the perpendicular bisector of CD with the line AB. This implies that $Y = Q$. Since Y lies on Γ it follows that $/CQD = /CYD = 90^\circ$.

Alternative 3 (Kathryn Zealand, Brisbane Girls Grammar School, QLD)

Let $/PCD = \alpha$ and let O be the midpoint of CD. Let X be the midpoint of the arc CD so that X and P are on opposite sides of the line CD. Let R be the intersection of QO and PC.



We now compute some angles as follows:

$$\begin{aligned}
\angle PQX &= \angle PQR \\
&= 180^\circ - \angle QPR - \angle QRP \\
&= 180^\circ - 45^\circ - \angle ORC \\
&= 135^\circ - (180^\circ - \angle ROC - \angle RCO) \\
&= 135^\circ - 180^\circ + 90^\circ + \alpha \\
&= 45^\circ + \alpha \\
&= \tfrac{1}{2}(90^\circ + 2\alpha) \\
&= \tfrac{1}{2}(\angle DOX + \angle POD) \\
&= \tfrac{1}{2}\angle POX.
\end{aligned}$$

Thus Q lies on the circumference of the circle centred at O with diameter CD. Hence $\angle CQD = 90^\circ$.

Solution 7 (Max Menzies, Sydney Grammar School, NSW)

Lemma. Let P be a point in the plane of triangle XYZ, then

$$2(PX + PY + PZ) \geq XY + YZ + ZX.$$

Proof. By three applications of the triangle inequality we have

$$\begin{aligned}
PX + PY &\geq XY \\
PY + PZ &\geq YZ \\
PZ + PX &\geq ZX.
\end{aligned}$$

Summing these three inequalities completes the proof of the lemma.

We shall apply the lemma to triangle $A_1A_2A_3$ for each point B_i where $i \in \{1, 2, 3\}$. Since

$$2(B_iA_1 + B_iA_2 + B_iA_3) \geq A_1A_2 + A_2A_3 + A_3A_1$$

holds for each $i \in \{1, 2, 3\}$, it follows that

$$\sum_{i=1}^{3} 2(B_iA_1 + B_iA_2 + B_iA_3) \geq 3(A_1A_2 + A_2A_3 + A_3A_1),$$

which implies

$$2q \geq 3(A_1A_2 + A_2A_3 + A_3A_1). \qquad (1)$$

If we apply the lemma in a similar way to triangle $B_1B_2B_3$ for each point A_i where $i \in \{1, 2, 3\}$ and sum the results, we similarly find

$$2q \geq 3(B_1B_2 + B_2B_3 + B_3B_1). \qquad (2)$$

Summing (1) and (2) yields $4q \geq 3p$, as required.

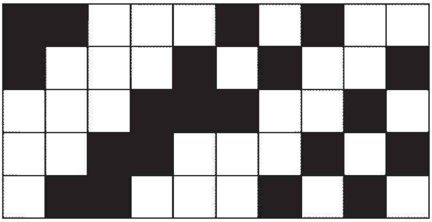
Solution 8 (Andrew Elvey Price, Brunswick Secondary College, VIC)

We define a *score* of a column to be the number of matching pairs of colours in that column. If all five squares of a column are the same colour then its score equals $\binom{5}{2} = 10$. If four are the same colour and one is the other colour, its score equals $\binom{4}{2} = 6$. If three are the same colour and two are the other colour, its score equals $\binom{3}{2} + \binom{2}{2} = 4$. Thus in all cases the score of any column is at least 4.

Let us sum the scores of all 2008 columns giving us a total score for the entire arrangement. This number is at least $4 \times 2008 = 8032$ and it represents the number of pairs of squares which are simultaneously in the same column and have the same colour.

For each pair of rows we count the number of columns with matching colours in those two rows. This yields a total score for each pair of rows. Since each pair of matching colours adds one to the score of exactly one pair of rows, the sum of the scores of the pairs of rows equals the score of the whole arrangement. Since there are $\binom{5}{2} = 10$ pairs of columns, by the pigeonhole principle at least one pair of columns has a score of at least $\left\lceil \frac{8032}{10} \right\rceil = 804$. Thus $N \geq 804$.

We now prove that $N \leq 804$. Consider the following colouring of a 5×10 chessboard.



Each pair of rows has a score of exactly 4. If we lay 201 copies of this configuration side by side making a 5×2010 array, each pair of rows has a score of $4 \times 201 = 804$. After removing any two columns we end up with a 5×2008 array in which each pair of rows has a score of at most 804. Thus $N \leq 804$.

Since $N \geq 804$ and $N \leq 804$ we conclude that $N = 804$.

RESULTS

Number of Candidates: 96

Gold

GARDAM Giles*	Hurlstone Agricultural High School	NSW	12
MENZIES Max*	Sydney Grammar School	NSW	12
ELVEY PRICE Andrew	Brunswick Secondary College	VIC	11
LIN Sen	James Ruse Agricultural High School	NSW	12
WONG Sampson	James Ruse Agricultural High School	NSW	10
ZHU Jonathan	Smith's Hill High School	NSW	12
CHONG Aaron	Doncaster Secondary College	VIC	10
LAW Stacey	James Ruse Agricultural High School	NSW	10
CHEUNG Paul	Sydney Technical High School	NSW	12
LO Irene	James Ruse Agricultural High School	NSW	12

Silver

BIUN John	Scotch College	VIC	12
ZHANG Bonnie	James Ruse Agricultural High School	NSW	12
ZEALAND Kathryn	Brisbane Girls Grammar School	QLD	12
MA Dana	Melbourne Girls Grammar School	VIC	11
GARDINER Sean	Sydney Grammar School	NSW	12
CAMERON Harry	Sydney Grammar School	NSW	12
WANG Zejun	Doncaster Secondary College	VIC	11
MARINOS Emmanuel	Sydney Grammar School	NSW	12
LI Adrian	Scotch College	VIC	11
LAM Kevin	Sydney Technical High School	NSW	10
KONG Jason	Christ Church Grammar School	WA	11

LIANG Alfred	Trinity Grammar School	VIC	10
PAN Kevin	James Ruse Agricultural High School	NSW	11
MAO Yitian	Baulkham Hills High School	NSW	12
NIRTHANAKUMARAN Deva Rajan	James Ruse Agricultural High School	NSW	11

Australian Team to participate in 2008 IMO at Madrid, Spain

Giles Gardam, Max Menzies, Andrew Elvey Price, Sampson Wong, Paul Cheung, Irene Lo.

2009

Problems

PAPER 1 Tuesday 10 February

1. Let ABC be an acute-angled triangle, and let P and Q be points on sides AC and BC, respectively, such that $APQB$ is a cyclic quadrilateral. Let R be the point such that PR is perpendicular to AC and QR is perpendicular to BC.

Prove that the line through C and R is perpendicular to AB.

2. Consider rectangular rooms whose side lengths are integers > 1. Tile the floors of such rooms with unit square tiles. *Edge tiles* are those next to the walls of the room (including corner tiles). The remaining tiles are called *interior tiles.*

A pair (R, S) of rooms (R may have the same dimensions as S) is *compatible* if

(i) the number of edge tiles in $R =$ the number of interior tiles in S and

(ii) the number of edge tiles in $S =$ the number of interior tiles in R.

For a compatible pair (R, S) let $m(R, S)$ be the minimal length of an edge occurring in R or S.

Determine the largest value $m(R, S)$ can take.

3. The polynomials $x^2 + x$ and $x^2 + 2$ are written on a white board. Sue is allowed to write on this board the sum, the difference or the product of any two polynomials already on the board. She repeats this process as many times as she likes.

Can Sue ever write the polynomial x on the board?

4. Let $a_1, a_2, \ldots, a_m$ be integers, where $m \geq 3$, and let

$$N = \frac{m(m+1)}{2}.$$

Prove that there is an integer k such that none of the integers $a_i + a_j - k$ is divisible by N, for all pairs of integers, where $1 \leq i \leq m$ and $1 \leq j \leq m$.

PAPER 2 Wednesday 11 February

5. A certain country has a finite number of towns, and all the distances between the towns are different. Each town is connected to its nearest neighbour by a straight road, and there are no other roads in the country.

(a) Prove that no two roads cross each other.

(b) Prove that there is no circuit within this road network.

(A *circuit* is a path consisting of different roads which begin and end at the same town.)

6. Determine all functions f that are defined for each positive integer, take positive integer values, and satisfy

$$\begin{array}{rcll} f(n) & = & n-9 & \text{if } n > 2009 \\ f(n) & = & f\left(f(n+1009)\right) & \text{if } n \leq 2009 \end{array}$$

7. Let I be the incentre of a triangle ABC in which $AC \neq BC$. Let Γ be the circle passing through A, I and B. Suppose Γ intersects the line AC at A and X and intersects the line BC at B and Y.

Show that $AX = BY$.

(The *incentre* of a triangle is the intersection of its angle bisectors.)

8. Let ABC be a triangle, and let X, Y, Z be points on the sides BC, CA, AB, respectively. Let T be the area of triangle XYZ and T_1, T_2, T_3 the areas of triangles AYZ, BZX, and CXY, respectively.

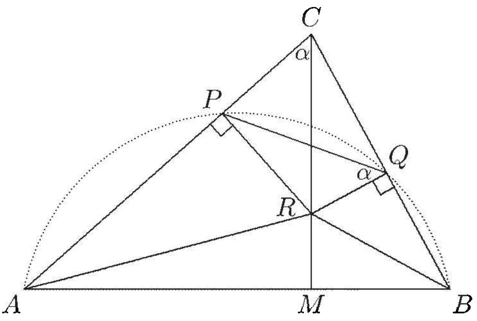
Prove that

$$\frac{3}{T} \leq \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3}.$$

Solution 1

Alternative 1 (Nancy Fu, Pymble Ladies' College, NSW)

Extend CR to intersect AB at point M. Let $\angle PCR = \alpha$.



Since $\angle CPR = \angle CQR = 90°$ it follows that $CPRQ$ is a cyclic quadrilateral. Thus $\angle PQR = \angle PCR = \alpha$. Thus $\angle PQB = 90° + \alpha$. However, we are given that $APQB$ is a cyclic quadrilateral. Thus $\angle PAB + \angle PQB = 180°$. Hence $\angle PAB = 90° - \alpha$.

Consider the angles in triangle ACM. We have that $\angle ACM = \angle PCR = \alpha$ and $\angle CAM = \angle PAB = 90° - \alpha$. Since the angle sum in triangle ACM is 180° we must have $\angle AMC = 90°$. Thus the line through C and R is perpendicular to AB.

Alternative 2 (Andrew Elvey Price, Brunswick Secondary College, VIC)

As in *Alternative 1* we extend CR to intersect AB at point M and note that $CPRQ$ is cylic. Thus

$$\begin{aligned} \angle PAM &= \angle PQC \quad (APQB \text{ cyclic}) \\ &= \angle PRC \quad (CPRQ \text{ cyclic}). \end{aligned}$$

Thus $APRM$ is cylic. Since $\angle APR = 90°$ it follows that $\angle AMR = 90°$. Hence CR is perpendicular to AB.

Alternative 3 (Angelo Di Pasquale)

By the Radical Axis Theorem. Let $F \in AB$ such that $RF \perp AB$. Then $APRF$ and $BQRF$ are cyclic. The three radical axes of circles $APRF$, $BQRF$ and $APQB$ are concurrent. Thus RF passes through the intersection of AP and BQ. Thus C, R, F are collinear. Hence $CR \perp AB$.

Solution 2

Alternative 1 (Timothy Large, Sydney Grammar School, NSW)

Let the dimensions of R and S be $a \times b$ and $c \times d$, respectively.

The number of exterior tiles of R is $2(a+b-2)$. The number of interior tiles of R is $ab-2(a+b-2)$. Similar results hold for room S.

The given conditions are equivalent to the following two equations.

$$\begin{array}{rcll} 2(a+b-2) & = & cd-2(c+d-2) & (1) \\ 2(c+d-2) & = & ab-2(a+b-2) & (2). \end{array}$$

Adding (1) and (2) together we have

$$\begin{array}{rrcll} & 4(a+b+c+d-4) & = & ab+cd & \\ \Leftrightarrow & (a-4)(b-4)+(c-4)(d-4) & = & 16 & (3). \end{array}$$

If $m(R,S) \geq 7$, then $a,b,c,d \geq 7$, implying LHS(3) ≥ 18. Contradiction.

Thus $m(R,S) \leq 6$. However if R and S are both 6×8 rooms then (1) and (2) are both satisfied and $m(R,S)=6$.

Thus 6 is the largest value that $m(R,S)$ can be.

Alternative 2 (Alfred Liang, Trinity Grammar School, VIC)

Let $R(E)$ and $R(I)$ denote the number of edge and interior tiles, respectively, of room R. $S(E)$ and $S(I)$ are defined similarly.

Suppose for a contradiction that $m(R,S) \geq 7$.

Let R have dimensions a and b. Then $R(I) > R(E)$ if and only if

$$\begin{array}{rrcl} & (a-2)(b-2) & > & 2a+2b-4 \\ \Leftrightarrow & (a-4)(b-4) & > & 8. \end{array}$$

However the last line is true because $a, b \geq 7$ by assumption.

Similarly $S(I) > S(E)$.

But now $R(I) > R(E) = S(I) > S(E) = R(I)$. Contradiction.

Hence $m(R,S) \leq 6$.

If R and S are both 6×8 rooms then $R(E) = R(I) = S(E) = S(I) = 24$. Thus $m(R,S) = 6$ is the maximum value sought.

Alternative 3 (Ian Wanless)

One of the rectangles (of dimension, say, m by n) in any compatible pair must have at least as many edge squares as it has middle squares. Suppose that $m \geq n \geq 7$. Then

$$\begin{array}{rcl} \text{number of edge squares} & = & 2(m+n-2) \\ & \geq & (m-2)(n-2) \\ & = & \text{number of middle squares.} \end{array}$$

So

$$2m \geq (m-4)(n-2) \geq 5(m-4),$$

which implies $3m \leq 20$. This contradicts $m \geq 7$. Since a pair of 6 by 8 rectangles is compatible, we see the answer is 6.

Alternative 4 (Problems Committee)

Let (A, B) be a compatible pair of rectangles for which the largest value of $m(A, B)$ is assumed. Let x, y be the lengths of the edges in A and u, v be the length of the edges in B. Without loss of generality, let $x \leq u \leq v \leq y$, thus $x = m(A, B)$.

Now $xy = uv$ since the total number of squares in either rectangle is the same. Further, the number of middle squares in A equals $(x-2)(y-2)$, and the number of edge squares in B equals $2u + 2v - 4$, so that $(x-2)(y-2) = 2u + 2v - 4$. Hence

$$xy = 2x + 2y + 2u + 2v - 8. \tag{1}$$

As x, u, v, y are positive integers, we have $xy \leq 8y - 8$, so that $(8-x)y \geq 8$ and therefore $x \leq 7$.

Suppose first that $x = 7$. From (1) we have $5y = 2u + 2v + 6 \leq 2y + 2y + 6 = 4y + 6$. Hence $y \leq 6$, which contradicts $7 = x \leq y$. We see immediately, however, that $x = 6$ is possible (with $y = 8$, $u = 6$, $v = 8$). Answer: $m(A, B) = 6$.

Solution 3

Alternative 1 (Stacey Law, James Ruse Agricultural High School, NSW)

Let $P(x) = x^2 + x$ and $Q(x) = x^2 + 2$. Note that $P(2) = Q(2) = 6$.

Suppose that $A(x)$ and $B(x)$ are any two polynomials on the board with the property that $A(2)$ and $B(2)$ are both divisible by 6. Then it follows that $A(2) + B(2)$, $A(2) - B(2)$ and $A(2)B(2)$ are also all divisible by 6. Thus any new polynomial, $R(x)$, formed will also have the property that $R(2)$ is divisible by 6.

It follows that *if* it were possible for Sue to write the polynomial $S(x) = x$ on the board, then $S(2)$ should be divisible by 6. This is a contradiction since $S(2) = 2$, which is not divisible by 6. Thus Sue can never write the polynomial x on the board.

Alternative 2 (Angelo Di Pasquale, Daniel Mathews and Ian Wanless)

It is easy to see that every polynomial Sue can write on the board must be of the form

$$a(x)(x^2 + x) + b(x)(x^2 + 2),$$

where $a(x)$ and $b(x)$ are polynomials with integer coefficients.

Suppose that

$$a(x)(x^2 + x) + b(x)(x^2 + 2) = x.$$

Then substituting in $x = -1$ we find

$$3b(-1) = -1.$$

However, this is impossible since 3 is not a factor of -1. Therefore Sue can never write x on the board.

Comment. It is possible to find exactly the set of polynomials that can be written on the board. The answer is: all polynomials of the form

$$(x - 2)g(x) + 6d,$$

where $g(x)$ is any integer polynomial and d is any integer.

To prove this let us use the substitution $x = y + 2$. Then the two given polynomials are $y^2 + 5y + 6$ and $y^2 + 4y + 6$. Note that these two polynomials generate the two polynomials y and 6. Conversely the polynomials y and 6 generate y^2+5y+6 and y^2+4y+6. Thus it suffices to find all polynomials generated by y and 6. This is straightforward. The answer is any polynomial of the form $yf(y)+6d$, where $f(y)$ is any integer polynomial and d is any integer. Thus the set of polynomials which Sue could generate are precisely those of the form $(x - 2)f(x - 2) + 6d = (x - 2)g(x) + 6d$, where $g(x) = f(x - 2)$ is any integer polynomial.

Solution 4

Alternative 1 (Andrew Elvey Price, Brunswick Secondary College, VIC)

We first note that there are at most $\frac{m(m+1)}{2} = N$ different sums $a_i + a_j$. Hence it suffices to show that there exist two different unordered pairs (p, q) and (r, s) such that

$$a_p + a_q \equiv a_r + a_s \pmod{N}. \quad (*)$$

Consider the multiset[1] S which consists of 0 and all expressions of form $a_i - a_j$ where $i \neq j$. Since there are m ways of choosing i and $m - 1$ ways of choosing j, we see that S has at most $m(m - 1) + 1$ different values. Furthermore, it is easy to verify that $m(m-1)+1 > \frac{m(m+1)}{2}$ for $m \geq 3$. Since there are only $N = \frac{m(m+1)}{2}$ different residues modulo N, it follows that two values of S are congruent modulo N. Two possibilities now arise.

---

[1] A multiset is like a set except that we allow an element to be a member of the multiset possibly more than once. For example, we are permitted to have $\{2, 2, 3, 3, 3\}$ as a multiset which has five elements, although if it were viewed as a set then it contains only the two elements 2 and 3.

If the two elements are 0 modulo N, then $a_i - a_j \equiv 0 \pmod N$ for $i \neq j$. In which case $a_1 + a_i \equiv a_1 + a_j \pmod N$.

If the two elements are not 0 modulo N, then $a_i - a_j \equiv a_k - a_l \pmod N$ where $(i,j) \neq (k,l)$ and $i \neq j$ and $k \neq l$, and so $a_i + a_l \equiv a_j + a_k \pmod N$.

Thus in both cases (*) is satisfied and the problem is solved.

Alternative 2 (Aaron Chong, Doncaster Secondary College, VIC)

We proceed by contradiction. Assume the conclusion of the question is false. Then every residue modulo N is represented by a pair $a_i + a_j$ $(i \leq j)$. Moreover, there are exactly $m + (m-1) + \cdots + 1 = \frac{m(m+1)}{2} = N$ such pairs. Hence each residue appears exactly once. Thus each a_i is distinct modulo N. Furthermore, we cannot have $a_p + a_q \equiv a_r + a_s \pmod N$ where (p,q) and (r,s) are distinct unordered pairs. Thus we have the following:

Lemma. If $a_p - a_r \equiv a_s - a_q \pmod N$ then either, $p = r$ and $q = s$, or $p = s$ and $q = r$.

Without loss of generality we may assume that

$$1 \leq a_1 < a_2 < \ldots < a_m \leq N < N + a_1.$$

Consider the m differences

$$a_2 - a_1,\ a_3 - a_2,\ \ldots,\ a_m - a_{m-1},\ N + a_1 - a_m.$$

From the lemma we see that all of these differences are distinct modulo N.

Since all m differences are distinct and positive, their sum is greater than or equal to $1 + 2 + \cdots + m = \frac{m(m+1)}{2}$. Thus

$$\begin{aligned} \tfrac{m(m+1)}{2} &\leq (a_2 - a_1) + \cdots + (a_m - a_{m-1}) + (N + a_1 - a_m) \\ &= N \\ &= \tfrac{m(m+1)}{2}. \end{aligned}$$

Hence equality holds throughout and the differences must be $1, 2, \ldots, m$ in some order. Thus there exists an index i such that

$$a_{i+1} - a_i \equiv 1 \pmod N.$$

However, $a_{i+2} - a_i$ cannot be any of $1, 2, \ldots, m$ modulo N otherwise we would have $a_{i+2} - a_i \equiv a_{j+1} - a_j \pmod N$ for some j and this would contradict the lemma. Hence $a_{i+2} - a_i \equiv k \pmod N$, where $k \geq m+1$. thus $a_{i+2} - a_{i+1} \equiv k - 1 \pmod N$ and $k - 1 \geq m$. Thus

$$a_{i+2} - a_{i+1} \equiv m \pmod N.$$

But a similar argument shows that $a_{i+1} - a_{i-1} \pmod N$ is also greater than m, so that $a_i - a_{i-1} \equiv m \pmod N$. However now we have

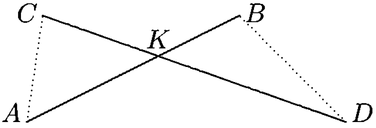
$$a_i - a_{i-1} \equiv a_{i+1} - a_i \pmod N.$$

This contradicts the lemma and finishes the proof.

Solution 5

Alternative 1 (Andrew Elvey Price, Brunswick Secondary College, VIC)

(a) Suppose two roads, AB and CD, cross each other. Let their intersection point be K. Without loss of generality A is the closest town to B and D is the closest town to C.



Thus

$$AB < BD \qquad \text{and} \qquad CD < AC.$$

So

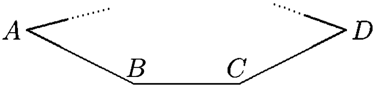
$$AB + CD < BD + AC.$$

However, from the triangle inequality we have

$$\begin{aligned} AB + CD &= AK + KB + CK + KD \\ &= (AK + CK) + (KB + KD) \\ &> AC + BD. \end{aligned}$$

Contradiction. Hence no two roads cross each other.

(b) Suppose that there is a circuit. Let BC be the longest road in the circuit and let A and D be towns connected to B and C, respectively, in the circuit. (N.B. A may happen to be the same as D, but this does not affect the proof.)



Then $BC > AB$ and $BC > CD$. So B is not the closest town to C and neither is C the closest town to B. So road BC does not exist. Contradiction. Hence there is no circuit in the road network.

Alternative 2 (part (b) only, from Problems Committee)

(b) Suppose there is a circuit, $ABC\dots YZ$ within the road network. Triangular or quadrilateral circuits can be quickly disposed of, so we can look at the general case.

For the distance between two cities C_1, C_2, we will simply write C_1C_2. Without loss of generality, we may assume that $ZA < AB$ (as it does not matter whether we travel through the circuit "clockwise" or "counter-clockwise"). Now, $AB < BC$ and so on. So we end up with

$$ZA < AB < BC < CD \dots < YZ < ZA,$$

which is a contradiction.

Alternative 3 (part (b) only, Ivan Guo)

(b) For any subset of k towns, there are at most $k-1$ road connections (as we join up the shortest distance, every other town can produce at most 1 more freeway). As this holds for any k, there can be no circuit.

Solution 6

Alternative 1 (Problems Committee)

If $1000 < n \leq 2009$, then

$$f(n) = f\left(f(n+1009)\right) = f(n+1000), \tag{1}$$

since $n + 1009 > 2009$. Next let $n \leq 1000$. Then

$$f(n) = f\left(f(n+1009)\right) = f\left(f(n+2009)\right),$$

by (1).

Now, $n + 1000 \leq 2000 < 2009$, so that $f(n+1000) = f\left(f(n+2009)\right)$. Hence $f(n) = f(n+1000)$ for all $n \leq 2009$.

Therefore the stated conditions are satisfied by exactly one function f, which is given by:

$$\begin{aligned} 3000 - k &= f(9-k) \quad \text{for } 0 \leq k \leq 8, \\ 3000 - k &= f(3009-k) \\ &= f(2009-k) \\ &= f(1009-k) \quad \text{for } 0 \leq k \leq 999, \\ \text{and} \quad f(n) &= n - 9 \quad \text{for } n > 2009. \end{aligned}$$

Alternative 2 (Dana Ma, Melbourne Girls Grammar School, VIC)
If $1 \leq n \leq 9$ we have

$$\begin{aligned} f(n) &= f(f(n+1009)) && \text{(since } n \leq 2009\text{)} \\ &= f(f(f(n+2018))) && \text{(since } n+1009 \leq 2009\text{)} \\ &= f(f(n+2009)) && \text{(since } n+2018 > 2009\text{)} \\ &= f(n+2000) && \text{(since } n+2009 > 2009\text{)} \\ &= f(f(n+3009)) && \text{(since } n+2000 \leq 2009\text{)} \\ &= f(n+3000) && \text{(since } n+3009 > 2009\text{)} \\ &= n+2991 && \text{(since } n+3000 > 2009\text{).} \end{aligned}$$

If $10 \leq n \leq 1000$ we have

$$\begin{aligned} f(n) &= f(f(n+1009)) && \text{(since } n \leq 2009\text{)} \\ &= f(f(f(n+2018))) && \text{(since } n+1009 \leq 2009\text{)} \\ &= f(f(n+2009)) && \text{(since } n+2018 > 2009\text{)} \\ &= f(n+2000) && \text{(since } n+2009 > 2009\text{)} \\ &= n+1991 && \text{(since } n+2000 > 2009\text{).} \end{aligned}$$

If $1001 \leq n \leq 1009$ we have

$$\begin{aligned} f(n) &= f(f(n+1009)) && \text{(since } n \leq 2009\text{)} \\ &= f(n+1000) && \text{(since } n+1009 > 2009\text{)} \\ &= f(f(n+2009)) && \text{(since } n+1000 \leq 2009\text{)} \\ &= f(n+2000) && \text{(since } n+2009 > 2009\text{)} \\ &= n+1991 && \text{(since } n+2000 > 2009\text{).} \end{aligned}$$

If $1010 \leq n \leq 2009$ we have

$$\begin{aligned} f(n) &= f(f(n+1009)) && \text{(since } n \leq 2009\text{)} \\ &= f(n+1000) && \text{(since } n+1009 > 2009\text{)} \\ &= n+991 && \text{(since } n+1000 > 2009\text{).} \end{aligned}$$

Also we are given that if $n > 2009$ then $f(n) = n - 9$. In summary the only solution is

$$f(n) = \begin{cases} n+2991 & \text{for } 1 \leq n \leq 9 \\ n+1991 & \text{for } 10 \leq n \leq 1009 \\ n+991 & \text{for } 1010 \leq n \leq 2009 \\ n-9 & \text{for } n > 2009. \end{cases}$$

For this solution to be complete, one must check that this answer satisfies the given conditions. This can be done by examining each of the cases $n \in [1,9]$, $[10,1000]$, $[1001,1009]$, $[1010,2009]$ and $[2010,\infty)$, separately. This is left as an exercise for the reader.

Alternative 3 (Angelo Di Pasquale).

$f(n) = n - 9$ for $n \geq 2010$.

$$\begin{aligned}
&f(2009) = f\left(f(2009 + 1009)\right) = f(3009) = 3000.\\
&f(2008) = f\left(f(2008 + 1009)\right) = f(3008) = 2999.\\
&\quad \dots\\
&f(1010) = f\left(f(1010 + 1009)\right) = f(2010) = 2001, \text{ then}\\
&f(1009) = f\left(f(1009 + 1009)\right) = f(2009) = 3000\\
&f(1008) = f\left(f(1008 + 1009)\right) = f(2008) = 2999\\
&\quad \dots\\
&f(1001) = f\left(f(1001 + 1009)\right) = f(2001) = 2992, \text{ then}\\
&f(1000) = f\left(f(2009)\right) = f(3000) = 2991\\
&f(999) = f\left(f(2008)\right) = f(2999) = 2990\\
&\quad \dots\\
&f(10) = f\left(f(1019)\right) = f(2010) = 2001, \text{ then}\\
&f(9) = f\left(f(1018)\right) = f(2009) = 3000\\
&f(8) = f\left(f(1017)\right) = f(2008) = 2999\\
&\quad \dots\\
&f(1) = f\left(f(1010)\right) = f(2001) = 2992.
\end{aligned}$$

The answer is

$$f(n) = \begin{cases} n + 2991 & \text{for } 1 \leq n \leq 9 \\ n + 1991 & \text{for } 10 \leq n \leq 1009 \\ n + 991 & \text{for } 1010 \leq n \leq 2009 \\ n - 9 & \text{for } 2010 \leq n. \end{cases}$$

To properly finish this we must verify this satisfies the equation. We must then check each of the intervals.

Solution 7

For this problem it was important to draw an accurate diagram. This would help anyone who was trying to solve this problem realize that one of X and Y lies on a non-extended side of the triangle, while the other lies on the extension of a side. Many contestants did not draw an accurate enough diagram and therefore thought that either X and Y both lay on the non-extended sides of the triangle or that they both lay on the extensions of the sides.

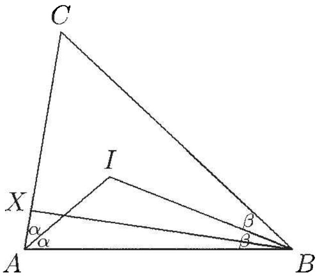
Trying to solve the problem using an impossible diagram is fraught with many difficulties. For example, many contestants "proved", using one of the impossible diagrams, that the triangle had to be isosceles, when the problem statement stipulated otherwise. Even in the cases where a contestant drew one of the correct diagrams, if the relative position of the points X and Y in the diagram was important in the argumentation of the proof, the solution was not complete unless the contestant proved that the correct relative positions were the only possible positions. This is called resolving the issue of diagram dependence in geometry proofs.

Alternative 1 (Stacey Law, James Ruse Agricultural High School, NSW)

Let $\angle CAB = 2\alpha$ and $\angle CBA = 2\beta$. Without loss of generality we may assume that $\alpha > \beta$.

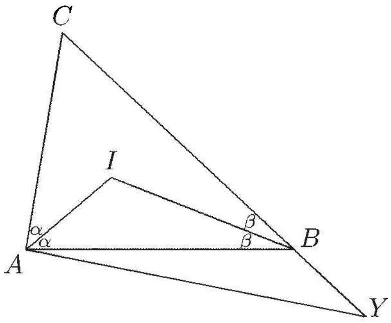
First we must take care of diagram dependence issues.

Suppose that X lies on the segment AC.



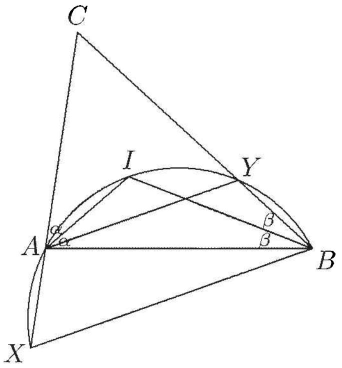
Then in cyclic quadrilateral $AXIB$ we have $\alpha = \angle XAI = \angle XBI < \angle ABI = \beta$. Contradiction. Thus X must lie on the extension of CA beyond A.

Suppose that Y lies on the extension of CB beyond B.



Then in cyclic quadrilateral $AIBY$ we have $\beta = \angle IBC = \angle IAY > \angle IAB = \alpha$. Contradiction. Thus Y lies on the segment BC.

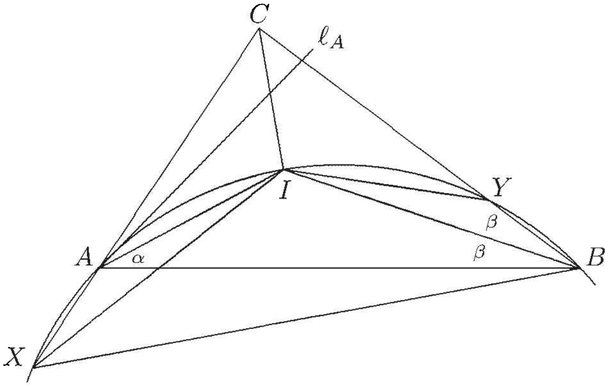
With the correct diagram as below we may complete the problem as follows.



Since $BYIAX$ is cyclic in that order we have $\angle IBX = \angle CAI = \alpha$. Hence $\angle ABX = \alpha - \beta$. Also $\angle IAY = \angle IBY = \beta$. Hence $\angle YAB = \alpha - \beta = \angle ABX$. Since equal angles are subtended by equal arcs we conclude that $AX = BY$.

Alternative 2 (Angelo Di Pasquale)

Let Γ be the circumcircle of triangle AIB. Let Γ intersect the line AC in A and X, and let the same circle intersect the line BC in B and Y. Let $\angle IAB = \angle IAC = \alpha$, and let $\angle IBA = \angle IBC = \beta$. Firstly there is a diagram dependence that is vital to discuss at the outset involving the positions of X and Y. Without loss of generality suppose that $\alpha > \beta$. This means that the tangent ℓ_A to Γ at A intersects the segment CI.



Thus the line AC intersects Γ on the extension of CA past A. Hence A lies between C and X on the line AC. By similar reasoning we deduce that Y lies between B and C on the line BC. Thus $XAIYB$ is a cyclic pentagon with vertices in that order around Γ.

Since IA and IY both subtend equal angles β at B, it follows that $AI = YI$. Also note that $\angle IBX = \angle IAC = \alpha = \angle IAB$ from cyclic quadrilateral $XAIB$, and thus $IX = IB$. We now observe that triangles AIX and YIB are congruent (AAS) because $\angle XAY = \angle IYB$ from $IX = IB$, $\angle AXI = \angle YBI$ from $AI = IY$ and $AI = IY$. Thus $AX = BY$, as required.

Alternative 3 (Angelo Di Pasquale)

Once we have dealt with diagram dependencies, we may angle chase as follows:

$$\begin{aligned}\angle BIY &= 180^\circ - \angle IBY - \angle IYB\\ &= 180^\circ - \beta - (180^\circ - \angle IAB)\\ &= \alpha - \beta.\end{aligned}$$

Also,

$$\begin{aligned}\angle ABX &= \angle IBX - \angle IBA\\ &= \angle IAC - \beta\\ &= \alpha - \beta.\end{aligned}$$

Thus $\angle ABX = \angle BIY$, which in turn implies that $AX = BY$.

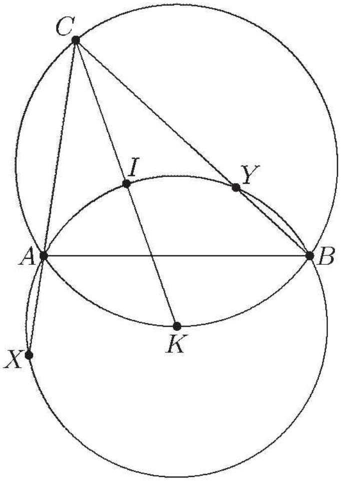
Alternative 4 (Aaron Chong, Doncaster Secondary College, VIC)

We discuss diagram dependency as in *Alternative 1*. We also use the same notation.

We have $\angle AYC = \angle BXC = 180^\circ - \angle AIB = \alpha + \beta$, from the angle sum in triangle AIB. Also we have $\angle YAC = \angle YAI + \angle IAC = \angle YBI + \alpha = \alpha + \beta = \angle AYC$. Hence $CA = CY$. Furthermore $\angle YXC = \angle YXA = \angle YBA = \angle CBA$. Thus triangles ABC and YXC are congruent (AAS). Hence $BC = XC$. Thus $AX = XC - AC = BC - CY = BY$.

Alternative 5 (Andrew Elvey Price, Brunswick Secondary College, VIC; Norman Do)

Let CI intersect the circumcircle of triangle ABC at K.



Then it is well known[2] that $KI = KB = KA$. Thus K is the centre of circle $BYIAX$. However since CI is the angle bisector at C, it follows that line CAX is the reflection of line CYB in CK. This implies that $AX = YB$. (It also implies that $CX = CB$.)

Alternative 6 (Chaitanya Rao)

By angle chase.

Using the figure of *Alternative 2*, we wish to show that arcs AX and BY are equal. Adding arc AY to both sides, we equivalently wish to show arcs AB and XY are equal, or that angles AXB and YBX are equal (i.e. that XCB is isosceles).

We have

$$\begin{aligned} \angle YXB &= \angle CBX \\ &= 180^\circ - \angle CXB - \angle XCB \\ &= 180^\circ - \angle ACB - \angle AXB \\ &= \angle CAB + \angle CBA - \angle AXB \\ &= 2(\angle IAB + \angle ABI) - \angle AXB \\ &= 2\angle AXB - \angle AXB \\ &= \angle AXB \end{aligned}$$

as required.

Alternative 7 (Ivan Guo)

Construct the perpendicular to CI through I, say it cuts AC at P. Now,

$$\angle PIA = \angle CIA - 90^\circ = 90^\circ - \angle ACI - \angle CAI = \angle IBA.$$

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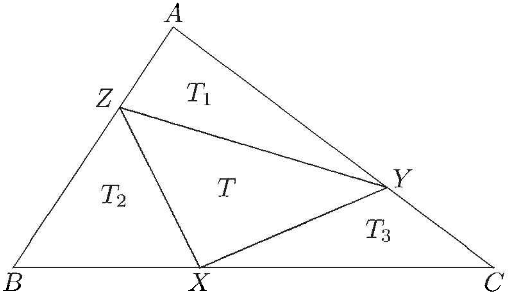
[2] The proof of this is a simple angle chase.

So, by the alternate segment theorem, PI is a tangent to circle AIB. Hence CI extends through the centre of that circle. As CI bisects $\angle ACB$, by (reflection) symmetry, the circle cuts out equal segments on AC, BC.

Solution 8

Alternative 1 (Problems Committee)

We use the following notation: if P is a point and QR is a line segment, then we will write $\delta(P, QR)$ for the perpendicular distance from P to QR.



We then have

$$\frac{T}{T_1}+1=\frac{T+T_1}{T_1}=\frac{\text{area}\,AXY}{\text{area}\,AYZ}+\frac{\text{area}\,AXZ}{\text{area}\,AYZ}=\frac{\delta(X,AC)}{\delta(Z,AC)}+\frac{\delta(X,AB)}{\delta(Y,AB)}.$$

Similarly, we obtain

$$\frac{T}{T_2}+1=\frac{\delta(Y,AB)}{\delta(X,AB)}+\frac{\delta(Y,BC)}{\delta(Z,BC)}.$$

and

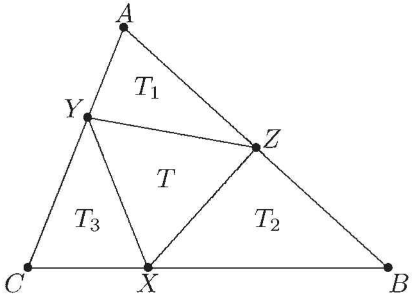
$$\frac{T}{T_3}+1=\frac{\delta(Z,BC)}{\delta(Y,BC)}+\frac{\delta(Z,AC)}{\delta(X,AC)}.$$

Adding these three equations leads to

$$\begin{aligned}3+\frac{T}{T_1}+\frac{T}{T_2}+\frac{T}{T_3} &= \frac{\delta(X,AC)}{\delta(Z,AC)}+\frac{\delta(X,AB)}{\delta(Y,AB)}+\frac{\delta(Y,AB)}{\delta(X,AB)}\\ &\quad+\frac{\delta(Y,BC)}{\delta(Z,BC)}+\frac{\delta(Z,BC)}{\delta(Y,BC)}+\frac{\delta(Z,AC)}{\delta(X,AC)}.\\ &= \left[\frac{\delta(X,AC)}{\delta(Z,AC)}+\frac{\delta(Z,AC)}{\delta(X,AC)}\right]\\ &\quad+\left[\frac{\delta(X,AB)}{\delta(Y,AB)}+\frac{\delta(Y,AB)}{\delta(X,AB)}\right]\\ &\quad+\left[\frac{\delta(Y,BC)}{\delta(Z,BC)}+\frac{\delta(Z,BC)}{\delta(Y,BC)}\right]\\ &\geq 6,\end{aligned}$$

since $x + \frac{1}{x} \geq 2$ if x is positive. Now the required inequality follows immediately.

Alternative 2 (Sampson Wong, James Ruse Agricultural High School, NSW)



Let $S = \text{Area}(ABC)$ and let $x = \frac{AY}{AC}$, $y = \frac{CX}{BC}$ and $z = \frac{BZ}{AB}$, where we know $0 \leq x, y, z \leq 1$. Then

$$\begin{aligned} \frac{T_1}{S} &= \frac{\frac{1}{2} AY \cdot AZ \sin \angle A}{\frac{1}{2} AC \cdot AB \sin \angle A} \\ &= x(1 - z). \end{aligned}$$

Similarly we have

$$\frac{T_2}{S} = z(1 - y) \quad \text{and} \quad \frac{T_3}{S} = y(1 - x).$$

Note also that since $T = S - T_1 - T_2 - T_3$ it follows that

$$\frac{T}{S} = 1 - x(1 - z) - z(1 - y) - y(1 - x).$$

For ease of manipulation we shall introduce the quantities $u = x(1 - z)$, $v = y(1 - x)$ and $w = z(1 - y)$.

The inequality to be proved becomes:

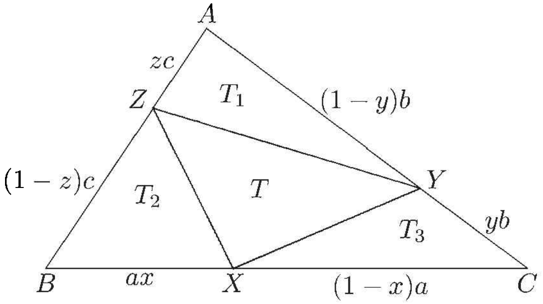
$$\begin{aligned}
\frac{3S}{T} &\le \frac{S}{T_1}+\frac{S}{T_2}+\frac{S}{T_3}\\
\Leftrightarrow \quad \frac{3}{1-u-v-w} &\le \frac{1}{u}+\frac{1}{v}+\frac{1}{w}\\
\Leftrightarrow \quad 3uvw &\le (1-u-v-w)(uv+vw+uw)\\
\Leftrightarrow \quad 6uvw &\le uv(1-u-v)+vw(1-v-w)\\
&\qquad +uw(1-u-w)\\
&= \sum_{\text{cyc}} uv(1-u-v)\\
\Leftrightarrow \quad 6 &\le \sum_{\text{cyc}} \frac{1-u-v}{w}\\
&= \sum_{\text{cyc}} \frac{1-x(1-z)-y(1-x)}{z(1-y)}\\
&= \sum_{\text{cyc}} \frac{(1-x)(1-y)+xz}{z(1-y)}\\
&= \sum_{\text{cyc}} \frac{1-x}{z}+\frac{x}{1-y}.
\end{aligned}$$

However, this follows directly from the AM-GM because

$$\begin{aligned}
\sum_{\text{cyc}} \frac{1-x}{z}+\frac{x}{1-y} &= \frac{1-x}{z}+\frac{x}{1-y}+\frac{1-y}{x}\\
&\qquad +\frac{y}{1-z}+\frac{1-z}{y}+\frac{z}{1-x}\\
&\ge 6\sqrt[6]{\frac{(1-x)x(1-y)y(1-z)z}{z(1-y)x(1-z)y(1-x)}}\\
&= 6.
\end{aligned}$$

Alternative 3 (Chaitanya Rao)

Let S be the area of ABC. Writing lengths of ABC as a, b, c and letting $BX = ax$, $XC = (1-x)a$, $CY = yb$, $YA = (1-y)b$, $AZ = zc$, $ZB = (1-z)c$



so that $0 < x, y, z < 1$, we have

$$\begin{aligned}
\frac{T}{T_1} &= \frac{(S - T_1 - T_2 - T_3)/S}{T_1/S} \\
&= \frac{1 - z(1-y) - x(1-z) - y(1-x)}{z(1-y)} \\
&= \frac{1 - x(1-z) - y(1-x)}{z(1-y)} - 1 \\
&= \frac{1 - x - y + xz + xy}{z(1-y)} - 1 \\
&= \frac{(1-x)(1-y) + xz}{z(1-y)} - 1 \\
&= \frac{1-x}{z} + \frac{x}{1-y} - 1.
\end{aligned}$$

Similarly we find

$$\frac{T}{T_2} = \frac{1-y}{x} + \frac{y}{1-z} - 1$$

and

$$\frac{T}{T_3} = \frac{1-z}{y} + \frac{z}{1-x} - 1.$$

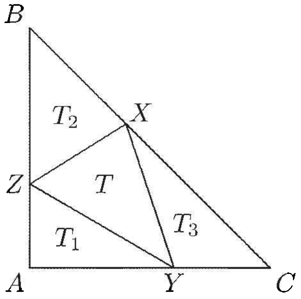
Summing:

$$\begin{aligned}
T\left(\frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}\right) &= \frac{x}{1-y}+\frac{1-y}{x}+\frac{z}{1-x}+\frac{1-x}{z} \\
&\quad +\frac{y}{1-z}+\frac{1-z}{y} \\
&\geq 2+2+2-3 \\
&= 3.
\end{aligned}$$

Dividing both sides by T gives the desired result.

Alternative 4 (Aaron Chong, Doncaster Secondary College, VIC)

An affine transformation preserves collinearity and ratios of corresponding areas. Since any triangle may be affinely transformed to any other triangle, it suffices to prove the inequality for the case where triangle ABC is a right isosceles triangle as shown in the diagram. The points have coordinates $A=(0,0)$, $B=(0,1)$, $C=(1,0)$, $X=(x,1-x)$, $Y=(1-y,0)$ and $Z=(0,z)$ where $0\leq x,y,z\leq 1$.



It is straightforward to calculate the areas

$$\begin{aligned}
T_1 &= \tfrac{1}{2}z(1-y) \\
T_2 &= \tfrac{1}{2}x(1-z) \\
T_3 &= \tfrac{1}{2}y(1-x) \\
T &= \text{Area}(ABC)-T_1-T_2-T_3 \\
&= \tfrac{1}{2}-\tfrac{1}{2}(z(1-y)+x(1-z)+y(1-x)) \\
&= \tfrac{1}{2}(1-x-y-z+xy+yz+zx) \\
&= \tfrac{1}{2}((1-x)(1-y)(1-z)+xyz).
\end{aligned}$$

Thus it suffices to prove

$$\frac{3}{(1-x)(1-y)(1-z)+xyz} \le \frac{1}{z(1-y)} + \frac{1}{x(1-z)} + \frac{1}{y(1-x)}.$$

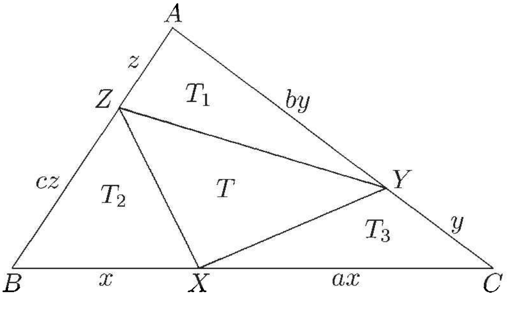
This is equivalent to proving

$$\begin{aligned} 3 &\le \sum_{\text{cyc}} \frac{(1-x)(1-y)(1-z)+xyz}{z(1-y)} \\ &= \sum_{\text{cyc}} \frac{(1-x)(1-z)}{z} + \frac{xy}{1-y} \\ &= \sum_{\text{cyc}} (1-x)\left(\frac{1}{z}-1\right) + x\left(\frac{1}{1-y}-1\right) \\ &= \sum_{\text{cyc}} \frac{1-x}{z} + \frac{x}{1-y} - 1 \\ \Leftrightarrow \quad 6 &\le \frac{1-x}{z} + \frac{1-y}{x} + \frac{1-z}{y} + \frac{x}{1-y} + \frac{y}{1-z} + \frac{z}{1-x} \\ &= \sum_{\text{cyc}} \frac{1-x}{z} + \frac{z}{1-x}. \end{aligned}$$

However, $\frac{1-x}{z} + \frac{z}{1-x} \ge 2$ from the AM-GM. Adding together the cyclic permutations of this inequality yields the required inequality above.

Alternative 5 (Angelo Di Pasquale)

Let S be the area of the triangle. Let $BX = x$, $XC = ax$, $CY = y$, $YA = by$, $AZ = z$, $ZB = cz$.



Note that

$$\frac{T_1}{s} = \frac{by(z\sin / A)}{[(b+1)y]\,[(c+1)z]\sin / A} = \frac{b}{(b+1)(c+1)}.$$

Similarly,

$$\frac{T_2}{s} = \frac{c}{(a+1)(c+1)}.$$

and

$$\frac{T_3}{s} = \frac{a}{(a+1)(b+1)}.$$

Further, $S = T_1 - T_2 - T_3$. Upon substitution, the inequality becomes

$$\frac{3abc(a+1)(b+1)(c+1)}{abc+1} \leq \frac{(b+1)(c+1)}{b} + \frac{(a+1)(c+1)}{c} + \frac{(a+1)(b+1)}{a}.$$

After some standard algebraic manipulation, using cyclic sum notation, the inequality becomes

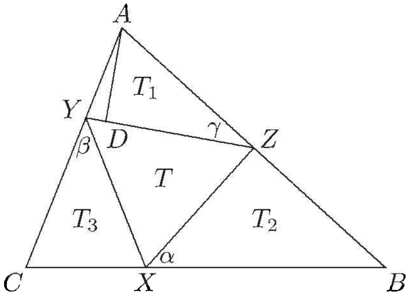
$$\sum_{cyc} a^3b^2c^2 + \sum_{cyc} ac^2 + \sum_{cyc} a^3b^2c + \sum_{cyc} ac \geq 2\sum_{cyc} a^2bc^2 + 2\sum_{cyc} a^2bc.$$

Yet this is a consequence of adding together the following two AM-GM inequalities:

$$\begin{aligned} \sum_{cyc} a^3b^2c^2 + \sum_{cyc} ac^2 = \sum_{cyc} \left(a^3b^2c^2 + ac^2\right) &\geq 2\sum_{cyc} a^2bc^2 \\ \sum_{cyc} a^3b^2c + \sum_{cyc} ac = \sum_{cyc} \left(a^3b^2c + ac\right) &\geq 2\sum_{cyc} a^2bc. \end{aligned}$$

Alternative 6 (Andrew Elvey Price, Brunswick Secondary College, VIC)

As in *Alternative 4* we note that the problem is invariant under an affine transformation. This time we transform the problem to the situation where triangle XYZ is equilateral and of unit side length.



Let $\alpha = \angle BXZ$, $\beta = \angle CYX$ and $\gamma = \angle AZY$. Consider triangle AYZ. From the angle sum at point Y we see that $\angle AYZ = 120° - \beta$. Let D

be the foot of the altitude from A to line YZ. Then

$$\begin{aligned}\frac{1}{T_1} &= \frac{2}{AD \cdot YZ}\\ &= \frac{2}{AD}\\ &= 2\left(\frac{YD}{AD} + \frac{DZ}{AD}\right)\\ &= 2(\cot(120^\circ - \beta) + \cot\gamma).\end{aligned}$$

Note that this formula is still correct if D lies on the extension of the line YZ rather than between Y and Z.

Similarly,

$$\frac{1}{T_2} = 2(\cot(120^\circ - \gamma) + \cot\alpha) \quad \text{and} \quad \frac{1}{T_3} = 2(\cot(120^\circ - \alpha) + \cot\beta).$$

Since $T = \frac{\sqrt{3}}{4}$, implying that $\frac{1}{T} = \frac{4\sqrt{3}}{3}$, the inequality to be proven becomes

$$\cot\alpha + \cot\beta + \cot\gamma + \cot(120^\circ - \alpha) + \cot(120^\circ - \beta) + \cot(120^\circ - \gamma) \geq 2\sqrt{3}.$$

However,

$$\begin{aligned}\cot\alpha + \cot(120^\circ - \alpha) &= \frac{\cos\alpha}{\sin\alpha} + \frac{\cos(120^\circ - \alpha)}{\sin(120^\circ - \alpha)}\\ &= \frac{\cos\alpha\sin(120^\circ - \alpha) + \cos(120^\circ - \alpha)\sin\alpha}{\sin\alpha\sin(120^\circ - \alpha)}\\ &= \frac{\sin 120^\circ}{\frac{1}{2}(\cos(120^\circ - 2\alpha) - \cos 120^\circ)}\\ &\geq \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}\left(1 - \left(-\frac{1}{2}\right)\right)}\\ &= \frac{2\sqrt{3}}{3}.\end{aligned}$$

Adding this together with the analogous results for $\cot\beta + \cot(120^\circ - \beta)$ and $\cot\gamma + \cot(120^\circ - \gamma)$ yields the desired inequality.

Comment. In the above algebraic manipulation we have made use of the product to sum trigonometric formula

$$\sin X \sin Y = \frac{1}{2}\left(\cos(X - Y) - \cos(X + Y)\right).$$

RESULTS

Number of Candidates: 101

Gold

CHONG Aaron*	Doncaster Secondary College	VIC	11
ELVEY PRICE Andrew*	Brunswick Secondary College	VIC	12
WONG Sampson	James Ruse Agricultural High School	NSW	11
LIANG Alfred	Trinity Grammar School	VIC	11
LAW Stacey	James Ruse Agricultural High School	NSW	11
MA Dana	Mebourne Girls Grammar School	VIC	12
LI Adrian	Scotch College	VIC	12
PARK Kiho	Mount Waverley Secondary College	VIC	11
LARGE Timothy	Sydney Grammar School	NSW	10
WANG Zejun	Doncaster Secondary College	VIC	12

Silver

VASAK David	Sydney Grammar School	NSW	11
LEI Yitao	All Saints Anglican School	QLD	11
RYBA Christopher	The Hutchins School	TAS	11
LU Colin	Melbourne Grammar School	VIC	10
STOKE Henry	Sydney Grammar School	NSW	11
LACY Thomas	Sydney Grammar School	NSW	12
WORMELL John	Sydney Boys High School	NSW	11
De SILVA Raveen	James Ruse Agricultural High School	NSW	11
LI Michael	Scotch College	VIC	10
YU Angel	Perth Modern School	WA	10
KONG Jason	Christ Church Grammar School	WA	12
LEE Vickie	Pymble Ladies' College	NSW	10
XU Ferris	All Saints College	WA	12
CHEUNG Justin	Sydney Grammar School	NSW	10
GIRDHAR Parth	James Ruse Agricultural High School	NSW	12
PAN Kevin	James Ruse Agricultural High School	NSW	12

Australian Team to participate in 2009 IMO at Bremen, Germany

Aaron Chong, Andrew Elvey Price, Sampson Wong, Alfred Liang, Stacey Law, Dana Ma.

2010

Problems

PAPER 1 Tuesday 09 February

1. Let $a_1, a_2, a_3, \ldots$ be real numbers satisfying

$$\begin{aligned} a_1 + a_2 &= 2010 \\ \text{and} \quad a_{n-1}a_{n+1} &= a_n, \quad \text{for } n = 2, 3, 4 \ldots \end{aligned}$$

Determine all possible values of $a_{2011} + a_{2012}$.

2. A *tiling* of an $m \times n$ chessboard is a complete covering of the board, without overlap, by 2×1 tiles. (A 2×1 tile covers exactly two squares of the chessboard and may be horizontal or vertical.)

(a) Prove that for every tiling of a 2010×4 chessboard there is a straight line dividing the board into two non-empty regions such that each tile lies completely in one of the regions.

(b) Prove that there is a tiling of a 2010×5 chessboard such that every straight line dividing the board into two non-empty regions crosses at least one tile.

3. Consider triangle ABC with $AB \neq AC$. Points P, Q, R and S are on the line through B and C.

P is the midpoint of BC.
AQ bisects $\angle BAC$.
AR and BC are perpendicular.
AS and AQ are perpendicular.

Prove that $PR \times QS = AB \times AC$.

4. A positive integer is said to be *square-free* if it is 1, is a prime or is the product of two or more different primes.

A prime p is called *good* if for all primes q less than p the remainder when p is divided by q is square-free.

Determine all good primes.

PAPER 2 Wednesday 10 February

5. Let K and L be concentric circles with radius r and s, respectively, and $r < s$. Let PA be a chord of K. Let BC be a chord of L passing through P and perpendicular to PA.

Find a formula, in terms of r and s, for $PA^2 + PB^2 + PC^2$.

6. Prove that

$$\begin{aligned} & \sqrt[3]{6+\sqrt[3]{845}+\sqrt[3]{325}}+\sqrt[3]{6+\sqrt[3]{847}+\sqrt[3]{539}} \\ = \; & \sqrt[3]{4+\sqrt[3]{245}+\sqrt[3]{175}}+\sqrt[3]{8+\sqrt[3]{1859}+\sqrt[3]{1573}}. \end{aligned}$$

7. Let a, b, c, d be integers satisfying $0 < a < b < c < d < 2010$.

Prove that there exists an integer e satisfying:

(i) $0 < e < 2010$,

(ii) e is a divisor of 2010, and

(iii) no two of a, b, c, d give the same remainder when divided by e.

8. Let O be the circumcentre and H the orthocentre of a triangle ABC with

$$\angle A < \angle B < \angle C < 90^\circ.$$

Prove that the incentre of triangle ABC lies inside triangle BHO.

Solution 1 (Yanning Xu, St Peter's College, SA)

Case 1. $a_1 = 0$. Then $a_2 = a_1a_3 = 0$. But this contradicts $a_1 + a_2 = 2010$.

Case 2. $a_2 = 0$. Then from $a_1 + a_2 = 2010$ we have $a_1 = 2010$.

Also $a_3 = a_2a_4 = 0a_4 = 0$. Then $a_4 = a_3a_5 = 0a_5 = 0$, and so on. In fact whenever we have $a_n = 0$, then we also have $a_{n+1} = a_na_{n+2} = 0a_{n+1} = 0$. Thus $a_n = 0$ for all $n \geq 2$. Hence our sequence is

$$\begin{aligned} a_1 &= 2010 \\ a_2 &= 0 \\ a_3 &= 0 \\ a_4 &= 0 \\ &\vdots \end{aligned}$$

It is easy to verify that this sequence satisfies the given properties.

Thus $a_{2011} + a_{2012} = 0 + 0 = 0$.

Case 3. $a_1 \neq 0$ and $a_2 \neq 0$. Write $a_1 = x$, $a_2 = y$ with $x, y \neq 0$. Note that $x + y = 2010$. Using $a_n = \frac{a_{n-1}}{a_{n-2}}$ for $n = 3, 4, \ldots$, we compute

$$\begin{aligned} a_1 &= x \\ a_2 &= y \\ a_3 &= \tfrac{y}{x} \\ a_4 &= \tfrac{1}{x} \\ a_5 &= \tfrac{1}{y} \\ a_6 &= \tfrac{x}{y} \\ a_7 &= x \\ a_8 &= y \\ &\vdots \end{aligned}$$

Note that $a_1 = a_7$ and $a_2 = a_8$. Since the value of a_n depends only on the values of a_{n-1} and a_{n-2} for $n \geq 3$, this means that we have entered a 6-cycle. Hence $a_n = a_{n+6}$ for all positive integers n. It is easy to verify that this 6-cycle satisfies the given properties for the sequence.

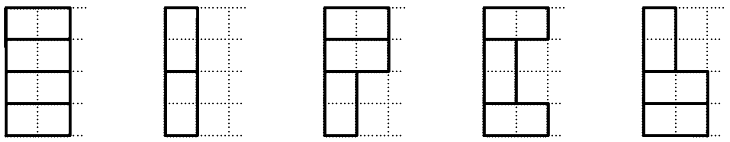
Thus since $2011 \equiv 1 \pmod 6$ and $2012 \equiv 2 \pmod 6$ it follows that $a_{2011} + a_{2012} = a_1 + a_2 = 2010$.

In conclusion we have shown that the only possible values for $a_{2011}+a_{2012}$ are 0 and 2010.

Solution 2

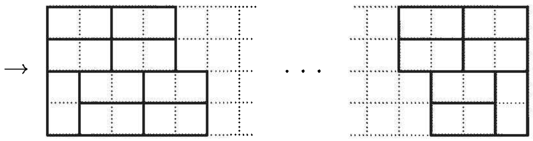
Alternative 1 (Victor Khou, James Ruse Agricultural High School, NSW)

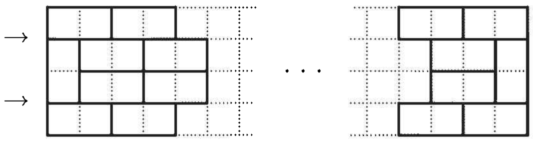
(a) Define a *fault line* to be a straight line which divides the chessboard into two non-empty regions such that each tile lies completely in one of the regions. Assume for the sake of contradiction, that we can tile a 2010×4 chessboard such that there is no fault line. With our chessboard oriented so that there are 4 rows and 2010 columns, there are five ways of filling up the first column as shown below.



Since we have assumed that there is no fault line, the first and second configurations are disallowed. Furthermore, the fifth configuration is just a mirror image of the third configuration. Hence we need only analyse the third and fourth configurations.

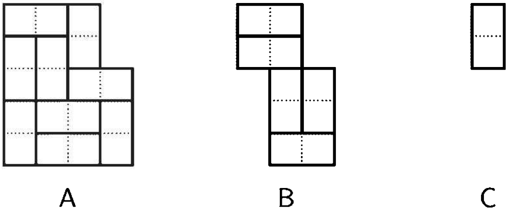
There is only one way to continue the third and fourth configurations across to the end while avoiding a vertical fault line. They are shown below. Note that in each case there is a horizontal fault line as indicated.



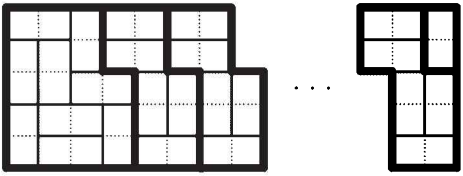


This is a contradiction. Hence it is impossible to tile a 2010×4 board without having a fault line.

(b) Consider the following three blocks of tiles.



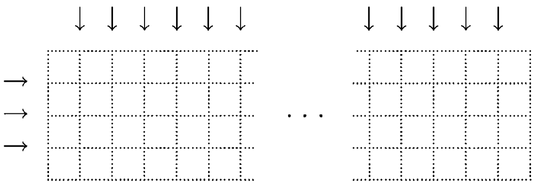
Paste together 1 copy of A with 1003 copies of B in the middle and finally 1 copy of C at the end as shown below.



This is a tiling of a 2010 × 5 chessboard which has no fault line.

Alternative 2 (Kiho Park, Mount Waverley Secondary College, VIC)

(a) We define a fault line as in the first alternative. We again orient the chessboard so that it has 4 rows and 2010 columns. Suppose that there is no fault line. Consider any of the 2009 vertical or 3 horizontal lines of the chessboard as indicated by an arrow below.

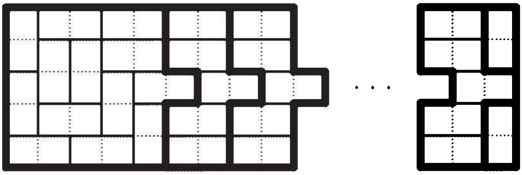


Note that there are an even number of squares which lie on either side of any such line.

Suppose for some such line ℓ, there are m tiles which lie entirely on one side of ℓ and n tiles which straddle ℓ. Then the total number of squares which are covered by tiles on one side of ℓ is equal to $2m + n$. Thus n must be even.

Since we have assumed that there is no fault line, this means that each of the indicated lines must be straddled by at least two tiles. Furthermore, no single tile can straddle more than one such line. This means that there are at least $2 \times (2009 + 3) = 4024$ tiles. However, the number of tiles is equal to $\frac{1}{2} \times 4 \times 2010 = 4020$, a contradiction.

(b) Here is a tiling of a 2010×5 chessboard which contains no fault line.



Alternative 3 ((a) only) (Sampson Wong, James Ruse Agricultural High School, NSW))

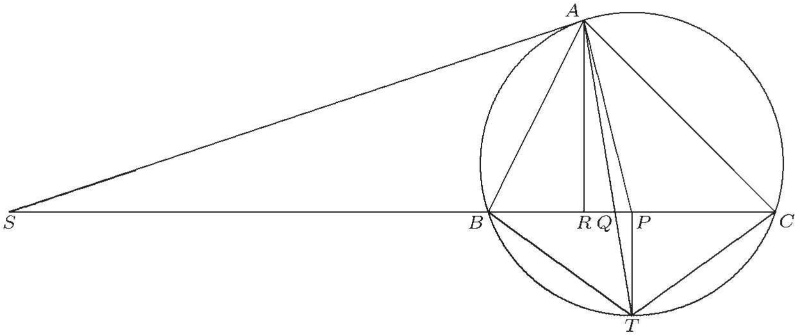
To rule out the 3 possible horizontal cuts there must be at least 3 vertical tiles. We can't have two vertical tiles at one end (otherwise there is a vertical cut) so there must be some vertical tile V which is not at either end. Now consider the possible vertical cuts on either flank of V. These must be blocked by two different horizontal tiles, with one tile bisected by the left hand cut and the other by the right hand cut.

Now take that left hand cut. It bisects just one tile, meaning the number s of squares to the left of it is odd. But s must be a multiple of 4, so we have a contradiction.

Solution 3

Alternative 1 (Problems Committee)

Without loss of generality, we may assume that $AC > AB$. Let the line through A and Q intersect the circumcircle of $\triangle ABC$ at T.



As $\angle BAT = \angle QAC$ (AQ is the bisector of $\angle BAC$) and $\angle ATB = \angle ACQ$ (both angles on the circumcircle of $\triangle ABC$ being subtended by AB), we see that $\triangle ATB$ and $\triangle ACQ$ are similar. Hence $AB : AQ = AT : AC$.

Thus $AB \times AC = AQ \times AT = AQ \times (AQ + QT) = (AQ)^2 + AQ \times AT$. As $AQ \perp AS$ and $AR \perp QS$, we have

$$AQ^2 = QR \times QS.$$

Moreover, $\angle BAT = \angle QAC$ implies that $BT = TC$, so that $PT \perp PS$. As $\angle AQS = \angle PQT$, the right-angled triangles AQS and PQT are similar, so that

$$AQ : QS = PQ : QT,$$

i.e.

$$AQ \times QT = PQ \times QS.$$

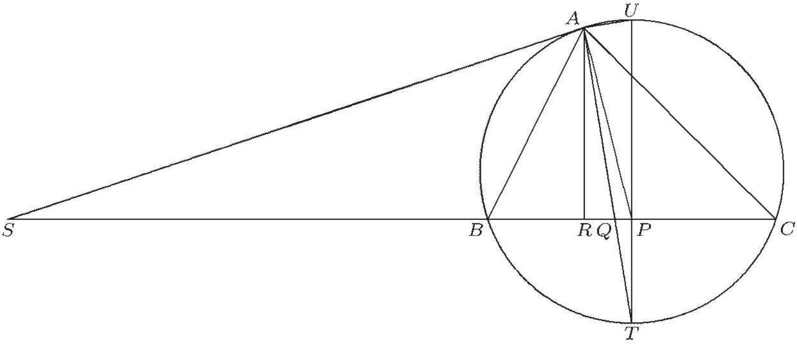
[The latter also follows directly from the power-of-a-point theorem applied to cyclic quadrilateral $SAPT$.]

Hence

$$\begin{aligned} AB \times AC &= AQ^2 + AQ \times QT \\ &= QR \times QS + AQ \times QT \\ &= QR \times QS + PQ \times QS \\ &= PR \times QS. \end{aligned}$$

Alternative 2 (Ivan Guo)

Define T as in Solution 1, and let U be the point opposite T on the circumcircle. A property of the internal and external angle bisectors at A is that they bisect the two arcs BTC and BAC, respectively. Thus points S, A and U are collinear.

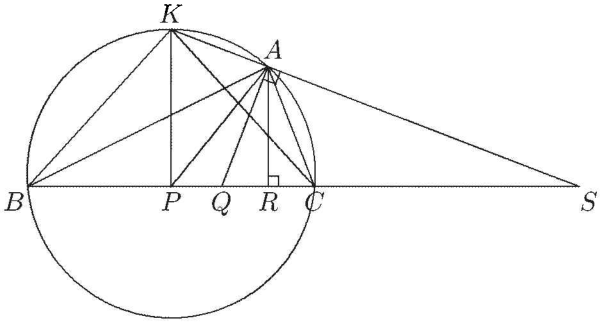


A simple angle chase shows that ASQ and ATU are similar triangles. Since AR is the height of $\triangle ASQ$ with base SQ and RP is the height of $\triangle ATU$ with base TU, it follows that $PR : AR = TU : SQ$, i.e. $PR \times SQ = AR \times TU$. If r denotes the radius of the circumcircle of $\triangle ABC$ and $\angle ABC = \beta$, then $TU = 2r$ and $AR = AB \sin \beta$. Hence

$$PR \times SQ = AR \times TU = AB \sin \beta \times 2r = AB \times AC.$$

Alternative 3 (Kiho Park, Mount Waverley Secondary College, VIC)

Let SA intersect the circumcircle for a second time at K.



Then note that $\angle KAB = 90^\circ - \angle BAQ = 90^\circ - CAQ = \angle CAS$. It then follows that $\angle KCB = \angle KAB = \angle CAS = \angle KBC$, where the last equality follows from cyclic quadrilateral $KBCA$. Thus $KB = KC$. Since P is the midpoint of BC this implies that $KP \perp BC$. Hence $KP \parallel AR$, from which it follows that

$$\frac{KA}{AS} = \frac{PR}{SR}.$$

Note also that triangles SAR and SQA are similar (AA). Hence

$$\frac{AS}{SR} = \frac{QS}{AS}.$$

Multiplying the above two equations together and rearranging yields

$$PR \cdot QS = KA \cdot AS. \quad (*)$$

Recall $\angle KAB = \angle CAS$ from earlier. It follows that $\angle BAS = \angle KAC$. Also $\angle AKC = \angle ABS$. Thus triangles KAC and BAS are similar (AA). Hence

$$\frac{KA}{AC} = \frac{AB}{AS}.$$

This may be rewritten as

$$KA \cdot AS = AB \cdot AC. \quad (**)$$

Thus it follows from (*) and (**) that

$$PR \cdot QS = AB \cdot AC.$$

Alternative 4 (Angelo Di Pasquale and Norman Do; Sampson Wong, James Ruse Agricultural High School, NSW)

We use the diagram of *Alternative 1.*

Let $a = BC$, $b = AC$, $c = AB$ and $\angle ACB = \gamma$. Then the angle bisector theorem yields $\frac{BQ}{CQ} = \frac{c}{b}$. This, together with $BQ + CQ = a$, yields $BQ = \frac{ac}{b+c}$. Similarly, with the external angle bisector AS, we see that $\frac{BS}{CS} = \frac{c}{b}$ and $CS - BS = a$, which implies $BS = \frac{ac}{b-c}$. Thus

$$QS = BS + BQ = \frac{2abc}{b^2 - c^2}.$$

Hence

$$\frac{AB.AC}{QS} = \frac{b^2 - c^2}{2a}.$$

Also $PR = RC - PC = b\cos\gamma - \frac{a}{2}$. However,

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab},$$

from the cosine rule in triangle ABC. Thus

$$PR = \frac{a^2 + b^2 - c^2}{2a} - \frac{a}{2} = \frac{b^2 - c^2}{2a} = \frac{AB.AC}{QS},$$

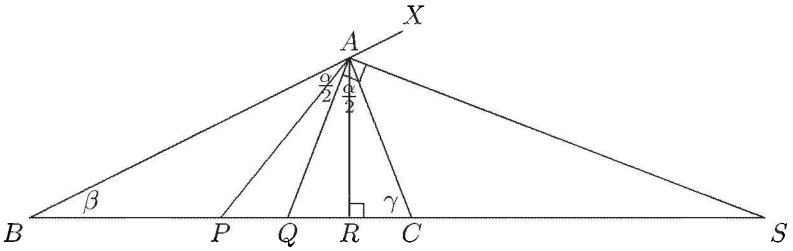
as required.

Alternative 5 (Yanning Xu, St Peter's College, SA)

Let $\angle BAC = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. Let R be the circumradius of triangle ABC. Without loss of generality we may assume that $\gamma > \beta$ so that C lies between B and S. Our strategy is to compute QS and PR in terms of α, β, γ and R.

We compute PR as follows:

$$\begin{aligned}
PR &= PC - RC \\
&= \tfrac{1}{2}BC - AC\cos\gamma \\
&= R\sin\alpha - 2R\sin\beta\cos\gamma \qquad \text{(extended sine rule)} \\
&= R(\sin(\beta+\gamma) - 2\sin\beta\cos\gamma) \qquad (\alpha+\beta+\gamma = 180^\circ) \\
&= R(\sin\beta\cos\gamma + \cos\beta\sin\gamma - 2\sin\beta\cos\gamma) \\
&= R(\cos\beta\sin\gamma - \sin\beta\cos\gamma) \\
&= R\sin(\gamma - \beta).
\end{aligned}$$



We now work on an expression for QS.

From $\angle CAS + \angle QAC = 90°$, we have $\angle CAS = 90° - \frac{\alpha}{2}$. Also since $\angle CAS + \angle CSA = \angle ACB$, we have $\angle CSA = \gamma + \frac{\alpha}{2} - 90° = \frac{\gamma}{2} - \frac{\beta}{2}$. The sine rule in triangles QAS and QAB yields the following:

$$\frac{QS}{AQ} = \frac{1}{\sin \angle CSA} = \frac{1}{\sin(\frac{\gamma-\beta}{2})}$$

$$\frac{AQ}{AB} = \frac{\sin\beta}{\sin(\beta+\frac{\alpha}{2})} = \frac{\frac{AC}{2R}}{\cos(\frac{\gamma-\beta}{2})}$$

where we have used $\sin\beta = \frac{AC}{2R}$ and $(\beta + \frac{\alpha}{2}) + (\frac{\gamma-\beta}{2}) = 90°$. Thus

$$QS = \frac{AB \cdot AC}{2R\sin(\frac{\gamma-\beta}{2})\cos(\frac{\gamma-\beta}{2})} = \frac{AB \cdot AC}{R\sin(\gamma-\beta)}.$$

It is now straightforward to compute that $PR \cdot QS = AB \cdot AC$, as required.

Alternative 6 (Malcolm Granville, Auckland Grammar School, NZ)

This solution uses complex numbers. We shall need to know the equation of a line which passes through two complex numbers u and v. All points on this line are given by $ru + (1 - r)v$ as r ranges over the real numbers. Thus the equation of the line is $z = ru + (1 - r)v = v + r(u - v)$. Thus $r = \frac{z-v}{u-v}$. Taking the complex conjugate of both sides yields $r = \frac{z-v}{u-v}$. Equating the two expressions for r yields

$$\frac{z-v}{u-v} = \frac{z-v}{u-v}.$$

This rearranges as

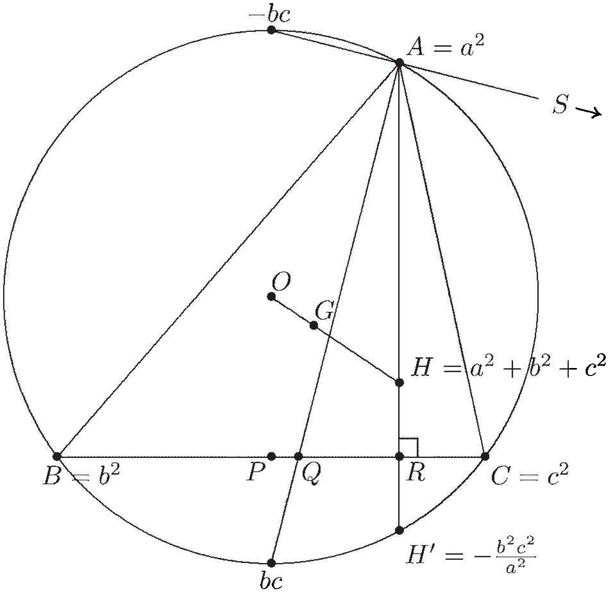
$$z(u - v) - z(u - v) = uv - uv.$$

However, for the problem at hand we will only ever need to consider when u, v lie on the unit circle. Thus $uu = vv = 1$ so that $u = \frac{1}{u}$ and $v = \frac{1}{v}$. Thus our equation simplifies to

$$z + zuv = u + v.$$

One final preparatory point. If u, v are on the unit circle, then the two quantities $\pm\sqrt{uv}$ represent the two midpoints of the arcs (the major arc and the minor arc) joining u to v in some order.

We now proceed to the given problem.



Without loss of generality we may assume that the circumcircle of triangle ABC is the unit circle in the complex plane. Let a^2, b^2, c^2 be the complex numbers representing points A, B, C, respectively.

Note that the internal and external angle bisectors at A are the lines AQ and AS, respectively. These intersect the unit circle at the midpoints of the two arcs which join B and C. These two points are thus bc and $-bc$ in some order. By choosing the value of b appropriately (there are two choices for b such that $B = b^2$) we may assume that AQ intersects the circumcircle at bc and AS intersects the circumcircle at $-bc$. Thus we have the equations given below for lines AQ and AS. We also have the equation for line BC.

$$\begin{aligned} AQ: \quad z + a^2bc\bar{z} &= a^2 + bc \\ AS: \quad z - a^2bc\bar{z} &= a^2 - bc \\ BC: \quad z + b^2c^2\bar{z} &= b^2 + c^2. \end{aligned}$$

We may now compute points P, Q, R, S.
Since P is the midpoint of BC it is given by

$$P = \frac{b^2 + c^2}{2}.$$

Point Q is the intersection of lines AQ and BC. Solving the system

$$\begin{aligned} z + a^2bcz &= a^2 + bc \\ z + b^2c^2z &= b^2 + c^2 \end{aligned}$$

yields

$$Q = \frac{a^2b^2 + a^2c^2 - a^2bc - b^2c^2}{a^2 - bc}.$$

Replacing b with $-b$ in the expression for Q, yields the expression for S

$$S = \frac{a^2b^2 + a^2c^2 + a^2bc - b^2c^2}{a^2 + bc}.$$

For point R, it is well known that the reflection of the orthocentre of a triangle in any of its three sides lies on the circumcircle of the triangle. Thus if H is the orthocentre and H' is the intersection of AH with the circumcircle, we have that R is the midpoint of HH'. It is well known that $\overrightarrow{OH} = 3\ \overrightarrow{OG}$, where G is the centroid and O is the circumcentre of triangle ABC. Since $G = \frac{a^2+b^2+c^2}{3}$, we have $H = a^2 + b^2 + c^2$. Furthermore if $H' = z$, then since A, H, H' are collinear, we must have $H' - A = r(H - A)$ for some real number r. Thus $z - a^2 = r(b^2 + c^2)$. Taking the complex conjugate and using $\overline{z} = \frac{1}{z}$, $\overline{a} = \frac{1}{a}$, $\overline{b} = \frac{1}{b}$, $\overline{c} = \frac{1}{a}$, we find $\frac{1}{z} - \frac{1}{a^2} = \frac{r(b^2+c^2)}{b^2c^2} = \frac{z-a^2}{b^2c^2}$. A little tidying up allows us to cancel a factor of $z - a^2$, since $z \neq a^2$, and thus find $z = -\frac{b^2c^2}{a^2}$. Since R is the midpoint of H and H' we deduce

$$R = \frac{1}{2}\left(a^2 + b^2 + c^2 - \frac{b^2c^2}{a^2}\right).$$

We can now calculate the values of PR and QS.

$$\begin{aligned} PR &= \left|\frac{1}{2}\left(a^2 + b^2 + c^2 - \frac{b^2c^2}{a^2}\right) - \frac{1}{2}(b^2 + c^2)\right| \\ &= \frac{|a^4 - b^2c^2|}{2|a^2|}. \\ QS &= \left|\frac{a^2bc + b^2c^2 - a^2b^2 - a^2c^2}{bc - a^2} - \frac{a^2bc - b^2c^2 + a^2b^2 + a^2c^2}{bc + a^2}\right| \\ &= \frac{2|bc(a^2 - b^2)(a^2 - c^2)|}{|a^4 - b^2c^2|}. \end{aligned}$$

Hence

$$\begin{aligned} PR\cdot QS &= \frac{|a^4-b^2c^2|}{2|a^2|}\cdot\frac{2|bc(a^2-b^2)(a^2-c^2)|}{|a^4-b^2c^2|} \\ &= \frac{|bc|}{|a^2|}\cdot|a^2-b^2|\cdot|a^2-c^2| \\ &= |a^2-b^2|\cdot|a^2-c^2| \qquad \text{(since } |a|=|b|=|c|=1) \\ &= AB\cdot AC. \end{aligned}$$

Solution 4

Alternative 1 (Nancy Fu, Pymble Ladies' College, NSW; Kevin McAvaney)

Of all the primes less than 20, we may check directly that 2, 3, 5, 7 and 13 are good. But the primes 11, 17 and 19 are bad, since $11 \equiv 4 \pmod 7$, $17 \equiv 4 \pmod{13}$ and $19 \equiv 4 \pmod 5$.

Suppose, for the sake of contradiction, that there is a prime $p > 20$ such that p is good. Consider $p-4$. If $p-4$ has a prime factor q which is greater than 4, then we would have $p \equiv 4 \pmod q$, a contradiction. Hence all prime factors of $p-4$ must be less than 4. Since $p-4$ is odd we must have $p-4 = 3^a$ for some positive integer a.

Consider $p-8$. If $p-8$ has a prime factor q which is greater than 8, then we would have $p \equiv 8 \pmod q$, a contradiction. Thus all prime factors of $p-8$ must be less than 8. Note that $p-8$ is odd, so that 2 is not a factor of $p-8$. From earlier $p-4=3^a$. So, $p-8=3^a-4$. Hence 3 is not a factor of $p-8$. Thus $p-8=5^{b_1}\cdot 7^{b_2}$, for non-negative integers b_1, b_2.

Consider $p-9$. By a similar argument as before we see that $p-9$ can have no prime factors greater than 9. Furthermore, $p-9=3^a-5$. Thus $p-9$ is not divisible by 3 or 5. Hence $p-9=2^{c_1}\cdot 7^{c_2}$.

Finally, consider $p-12$. By a similar argument $p-12$ has no prime factors greater than 12. Since $p-12=3^a-8$ we see that $p-12$ is not divisible by 2 or 3. Thus $p-12=5^{d_1}\cdot 7^{d_2}\cdot 11^{d_3}$ for non-negative integers d_1, d_2, d_3.

To summarise so far, if $p > 20$ is a good prime then

$$\begin{aligned} p-4 &= 3^a \\ p-8 &= 5^{b_1}\cdot 7^{b_2} \\ p-9 &= 2^{c_1}\cdot 7^{c_2} \\ p-12 &= 5^{d_1}\cdot 7^{d_2}\cdot 11^{d_3}. \end{aligned}$$

Suppose that $c_2 = 0$, thus $p - 9 = 3^a - 5 = 2^{c_1}$. Since $p > 20$ we have $c_1 > 2$. Hence $3^a \equiv 5 \pmod 8$. However the powers of 3 modulo 8 cycle as $1, 3, 1, 3, \ldots$, a contradiction. Hence $c_2 \geq 1$. Thus $p - 9$ is divisible by 7. Hence neither of $p - 8$ or $p - 12$ are divisible by 7. Thus $b_2 = d_2 = 0$. Furthermore, since $p - 8 = 5^{b_1}$, it necessarily follows that $b_1 > 0$ and so $p - 8$ is divisible by 5. But then $p - 12$ cannot be divisible by 5. Thus $d_1 = 0$. Hence

$$p - 12 = 3^a - 8 = 11^{d_3}$$

with $d_3 > 0$. Hence $3^a \equiv 8 \pmod{11}$. However, powers of 3 modulo 11 cycle as $1, 3, 9, 5, 4, 1, 3, 9, 5, 4, \ldots$, a contradiction. This contradiction establishes that there are no more good primes.

Alternative 2 (Alfred Liang, Trinity Grammar School, VIC)

As in the first alternative we find that 2, 3, 5, 7 and 13 are good primes, while 11 is not a good prime. We also deduce the following equations in much the same way.

$$\begin{aligned} p - 4 &= 3^a \\ p - 8 &= 5^{b_1} \cdot 7^{b_2} \\ p - 9 &= 2^{c_1} \cdot 7^{c_2}. \end{aligned}$$

Since $p-8$ and $p-9$ are consecutive integers, they cannot both be divisible by 7. It follows that at least one of b_2, c_2 is equal to zero. Subtracting the equations for $p - 8$ and $p - 9$ we have

$$5^{b_1} \cdot 7^{b_2} = 2^{c_1} \cdot 7^{c_2} + 1.$$

It suffices to discuss the following three cases.

Case 1. $b_2 = c_2 = 0$. In this case we have $5^{b_1} - 2^{c_1} = 1$. However by Catalan's Theorem[3] this is impossible.

Case 2. $b_2 = 0$, $c_2 \neq 0$. In this case we have $5^{b_1} - 1 = 2^{c_1} \cdot 7^{c_2}$. Note that $2^{c_1} \cdot 7^{c_2}$ is not divisible by 3. Hence, $5^{b_1} \not\equiv 1 \pmod 3$. However, 5^i modulo 3 cycles as $1, 2, 1, 2, \ldots$ for $i = 0, 1, 2, 3, \ldots$. Thus b_1 is odd.

Now consider the same equation modulo 7. We have 5^i modulo 7 cycles as

$$1, 5, 4, 6, 2, 3, 1, \ldots$$

for $i = 0, 1, 2, 3, 4, 5, 6, \ldots$. Considering only the odd values of i we see that LHS $\equiv 4$, 5 or $2 \pmod 7$. But RHS $\equiv 0 \pmod 7$. Thus there is no solution in this case.

Case 3. $b_2 \neq 0$, $c_2 = 0$. In this case we have $5^{b_1} \cdot 7^{b_2} = 2^{c_1} + 1$. Thus $2^{c_1} \equiv 6 \pmod 7$. However, 2^i modulo 7 cycles as $1, 2, 4, 1, \ldots$ for $i = 0, 1, 2, 3, \ldots$. Thus there is no solution in this case either.

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[3] *Catalan's Theorem* states that if the difference between two non-trivial perfect powers is equal to 1, then those perfect powers are 3^2 and 2^3, respectively. This theorem remained a conjecture until 2002 when it was finally proved by Preda Mihăilescu.

Comment. While this proof is complete, here is a way of dealing with case 1 if one wishes to avoid invoking Catalan's Theorem. First note that there are no solutions to the equation $5^{b_1} - 2^{c_1} = 1$ for $c_1 < 3$. For $c_1 \geq 3$ consider the equation modulo 8. We need

$$5^{b_1} \equiv 1 \pmod 8.$$

This implies that b_1 is even. Write $b_1 = 2b$ and rearrange the equation to get

$$(5^b + 1)(5^b - 1) = 2^{c_1}.$$

Hence both factors $5^b + 1$ and $5^b - 1$ must be powers of 2. Furthermore their difference is equal to 2. Thus $5^b + 1 = 4$ and $5^b - 1 = 2$. This is impossible.

Alternative 3 (Malcolm Granville, Auckland Grammar School, NZ)

As in the first two alternatives we verify that 2, 3, 5, 7 and 13 are the only good primes less than 14. We also derive the equations

$$\begin{aligned} p-4 &= 3^a \\ p-8 &= 5^{b_1} \cdot 7^{b_2} \\ p-9 &= 2^{c_1} \cdot 7^{c_2}. \end{aligned}$$

Case 1. $p - 8$ is not divisible by 7. Thus $b_2 = 0$. Subtracting the equations for $p - 4$ and $p - 8$ yields

$$3^a - 4 = 5^{b_1}.$$

By considering this equation modulo 4 we find that $(-1)^a \equiv 1 \pmod 4$. Thus a is even. Write $a = 2k$ for some positive integer k. Hence

$$(3^k - 2)(3^k + 2) = 5^{b_1}.$$

Hence both factors $3^k - 2$ and $3^k + 2$ are powers of 5. However, since they differ by 4, they must be 1 and 5, respectively. Thus $k = 1$, $a = 2$ and $p = 13$ which has already been discussed.

Case 2. $p - 8$ is divisible by 7. Thus $p - 9$ is not divisible by 7 implying that $c_2 = 0$. Subtracting the equations for $p - 4$ and $p - 9$ yields

$$3^a = 2^{c_1} + 5.$$

Considering this equation modulo 4 yields $(-1)^a \equiv 1 \pmod 4$. Thus a is even. Write $a = 2k$ for some positive integer k. Considering the same equation modulo 3 yields $(-1)^{c_1} \equiv 1 \pmod 3$. Thus c_1 is also even. Write $c_1 = 2\ell$ for some positive integer ℓ. As a consequence we find the following two expressions for p

$$p = 2^{2k} + 3^2 \qquad \text{and} \qquad p = 3^{2\ell} + 2^2.$$

Thus unless $k = \ell = 1$, which is precisely the case $p = 13$, we see that p has two distinct representations as a sum of squares. By a theorem of Fermat[4] this would imply that p is composite, a contradiction.

Comment. While this proof is complete, it is possible to finish the proof without using Fermat's result. Subtract the two expressions for p and factorize to get $(3^\ell + 2^k)(3^\ell - 2^k) = 5$. Thus $3^\ell + 2^k = 5$ and $3^\ell - 2^k = 1$. Summing these last two equations yields $\ell = 1$, then $k = 1$ and $p = 13$.

Alternative 4 (Norman Do)

We verify by inspection that 2, 3, 5, 7, 13 are good primes. Now let p be a good prime with $p > 7$.

Suppose that $p - 4$ has a prime factor q larger than 4, then the remainder when p is divided by q equals 4, which is a contradiction. Since $p - 4$ is odd, it follows that $p - 4$ is a power of 3, and as we assumed that $p > 7$, we must have $p - 4 = 3^\ell$ with $\ell \geq 2$.

Similarly, $p - 8$ has no prime factor larger than 8, so that $3^\ell - 4 = p - 8 = 5^{b_1} 7^{b_2}$ (noting that 3 does not divide $3^\ell - 4$).

We may assume that ℓ is at least three. Our goal is to show that there are no solutions. If $b_1 > 0$, then we can read the equation modulo 7 to obtain $3^\ell \equiv 4 \pmod 7$, which implies that $\ell \equiv 2 \pmod 4$. If $b_2 > 0$, then we can read the equation modulo 7 to obtain $3^\ell \equiv 4 \pmod 5$ which implies that $\ell \equiv 4 \pmod 6$. Either way, $\ell = 2c$ is even, so $(3^c - 2)(3^c + 2) = 5^{b_1} 7^{b_2}$.

Clearly, $\gcd(3^c - 2, 3^c + 2) = 1$, so we have one of two cases.

Case 1: $3^c + 2 = 5^{b_1}$ and $3^c - 2 = 7^{b_2}$.

Reading the first equation modulo 5, we obtain $3^c \equiv 3 \pmod 5$, which implies that $c \equiv 1 \pmod 4$. Reading the second equation modulo 7, we obtain $3^c \equiv 2 \pmod 7$, which implies that $c \equiv 2 \pmod 6$. However, this implies that c is simultaneously odd and even – a contradiction.

Case 2: $3^c + 2 = 7^{b_2}$ and $3^c - 2 = 5^{b_1}$.

Reading the first equation modulo 3, we obtain $7^{b_2} \equiv 2 \pmod 3$, which is impossible since $7^{b_2} \equiv 1^{b_2} = 1 \pmod 3$.

Solution 5

Alternative 1 (Patrick He, Scotch College, VIC)

Let O be the common centre of circles K and L. Let M and N be the midpoints of BC and PA, respectively. By symmetry $OM \perp BC$ and $ON \perp PA$. Thus $OMPN$ is a rectangle. Hence $OM = \frac{1}{2} PA$.

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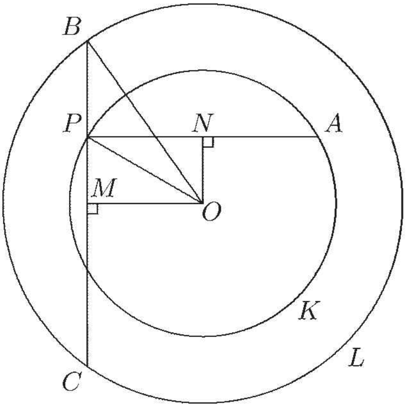
[4] Fermat proved that primes of form $4k + 3$ can not be represented as a sum of two perfect squares but all other primes can be represented as a sum of two perfect squares. Furthermore, such a representation is unique. Thus any number which has more than one representation as a sum of two perfect squares must be composite.



By Pythagoras we have

$$BM^2 + OM^2 = OB^2.$$

However, since $BM = \frac{1}{2}BC$, $OM = \frac{1}{2}AP$ and $OB = s$ it follows that

$$\begin{array}{rrcl} & \frac{1}{4}BC^2 + \frac{1}{4}PA^2 & = & s^2 \\ \Rightarrow & (PB + PC)^2 + PA^2 & = & 4s^2 \\ \Rightarrow & PB^2 + PC^2 + 2PB \cdot PC + PA^2 & = & 4s^2. \qquad (*) \end{array}$$

However, by considering the power of P with respect to circle L we have

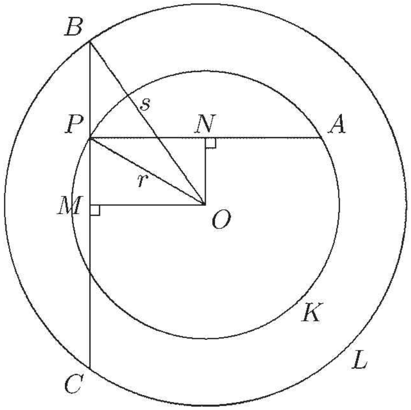
$$PB \cdot PC = s^2 - r^2.$$

Substituting this into (*) and rearranging yields

$$PA^2 + PB^2 + PC^2 = 2r^2 + 2s^2.$$

Alternative 2 (Aaron Chong, Doncaster Secondary College, VIC)
Let O be the common centre of circles K and L. Let M and N be the midpoints of BC and PA, respectively. Thus $OM \perp BC$ and $ON \perp PA$. Hence $OMPN$ is a rectangle. It follows that $OM = \frac{1}{2}PA$.
We have the following three expressions for PA^2, PB^2 and PC^2:

$$\begin{aligned} PA^2 &= 4OM^2 \\[1em] PB^2 &= (MB - MP)^2 \\ &= MB^2 - 2MB \cdot MP + MP^2 \\[1em] PC^2 &= (MC + MP)^2 \\ &= MC^2 + 2MC \cdot MP + MP^2. \end{aligned}$$



Adding the expressions together and noting that $MB = MC$, yields

$$\begin{aligned} PA^2 + PB^2 + PC^2 &= 4OM^2 + 2MP^2 + 2MB^2 \\ &= 2(OM^2 + MP^2) + 2(OM^2 + MB^2) \\ &= 2r^2 + 2s^2, \end{aligned}$$

where the last line follows from Pythagoras' Theorem applied to triangles OMP and OMB.

Alternative 3 (Victor Khou, James Ruse Agricultural High School, NSW)

Let O be the centre of circles K and L. Let R be the second intersection point of PC with circle K. Since $\angle APR = 90°$ it follows that AR is a diameter of circle K. Thus O is the midpoint of AR.

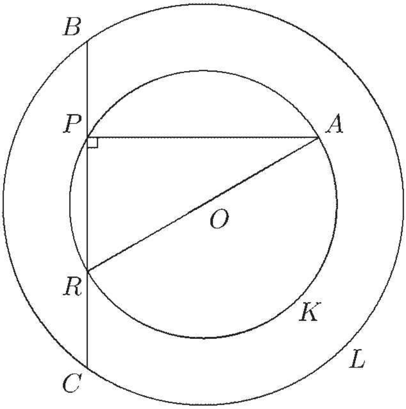
We may thus compute

$$\begin{aligned} PA^2 &= RA^2 - PR^2 \\ &= (2r)^2 - (RB - PB)^2 \\ &= 4r^2 - RB^2 - PB^2 + 2RB \cdot PB \\ &= 4r^2 - PC^2 - PB^2 + 2BP \cdot BR \qquad (*) \end{aligned}$$

where $RB = PC$ follows from symmetry about the perpendicular bisector of BC.

Finally we apply power of a point to the point B with respect to circle K to deduce $BP \cdot BR = BO^2 - r^2 = s^2 - r^2$. Upon substitution into (*) and rearranging we find

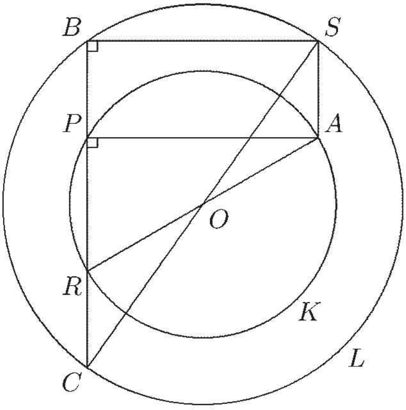
$$PA^2 + PB^2 + PC^2 = 2r^2 + 2s^2.$$



Alternative 4 (Michael Li, Scotch College, VIC)

As in the third alternative let O be the centre of circles K and L. Let R be the second intersection point of PC with circle K. Since $\angle APR = 90°$ it follows that AR is a diameter of circle K. Thus O is the midpoint of AR.

Let S be the second point of intersection of line CO with circle L. Since SC is a diameter of circle L, we have $SB \perp BC$. Furthermore, segment SA is the result of applying a half turn to segment CR about point O. Hence $SA \parallel CR$. Thus $SA \perp PA$. It follows that $PASB$ is a rectangle. Thus $PA = BS$ and $CR = SA = PB$.



We apply Pythagoras' Theorem to triangle RAP to obtain

$$\begin{aligned} PA^2 + PR^2 &= AR^2 \\ \Rightarrow \quad PA^2 + (PC - CR)^2 &= 4r^2 \\ \Rightarrow \quad PA^2 + (PC - PB)^2 &= 4r^2 \\ \Rightarrow \quad PA^2 + PB^2 + PC^2 &= 4r^2 + 2PC \cdot PB. \qquad (*) \end{aligned}$$

We also apply Pythagoras' Theorem to triangle BCS to obtain

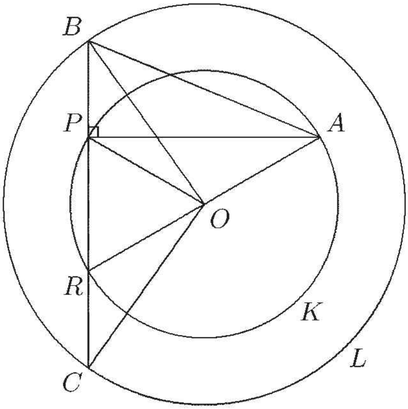
$$\begin{aligned} BS^2 + BC^2 &= SC^2 \\ \Rightarrow \quad PA^2 + (PC + PB)^2 &= 4s^2 \\ \Rightarrow \quad PA^2 + PB^2 + PC^2 &= 4s^2 - 2PC \cdot PB. \qquad (**) \end{aligned}$$

Adding equations (*) and (**) together yields

$$PA^2 + PB^2 + PC^2 = 2r^2 + 2s^2.$$

Alternative 5 (Robert Zhang, Auckland Grammar School, New Zealand)
Let O be the centre of circles K and L. Let R be the second intersection point of PC with circle K. Since $\angle APR = 90^\circ$ it follows that AR is a diameter of circle K. Thus O is the midpoint of AR.

Furthermore triangles OBP and OCR are congruent (SAS). This follows from $\angle BOP = \angle OPR - \angle OBP = \angle ORP - \angle OCR = \angle COR$, $OB = OC$ and $OP = OR$. Hence $PB = RC$ and so $PC = BR$.



From Pythagoras' Theorem we have

$$PA^2 + PB^2 = AB^2.$$

Thus

$$PA^2 + PB^2 + PC^2 = AB^2 + BR^2.$$

Since O is the midpoint of AR we may apply Apollonius' Theorem to triangle ABR to get

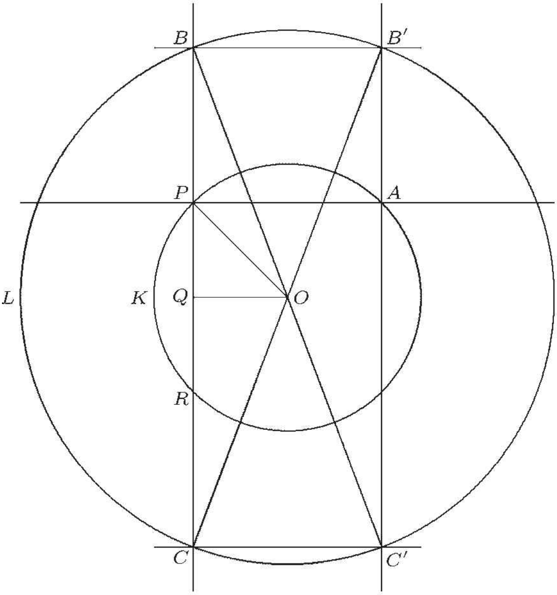
$$\begin{aligned} AB^2 + BR^2 &= 2OB^2 + 2(\tfrac{1}{2}AR)^2 \\ &= 2s^2 + 2r^2. \end{aligned}$$

Thus

$$PA^2 + PB^2 + PC^2 = 2r^2 + 2s^2.$$

Alternative 6 (Problems Committee)

Let the line through A which is parallel to BC intersect L in B' and C' (so that BB' is parallel to PA).



Then the quadrilaterals $APBB'$ and $APCC'$ are rectangles, so that, by Pythagoras,

$$(PB + PC)^2 + PA^2 = (PB + PC)^2 + BB'^2 = 4s^2,$$

i.e.

$$PB^2 + PC^2 + PA^2 = 4s^2 - 2PB.PC.$$

By the secant-chord theorem,

$$PB.PC = (s + r)(s - r) = s^2 - r^2,$$

implying that

$$PB^2 + PC^2 + PA^2 = 2(s^2 + r^2).$$

Alternative 7 (Chaitanya Rao)

Let R be the other intersection of BC with K ($P = R$ in the case of tangency). Wlog assume $PC \geq PB$ as in the figure given in Solution 1. Then

$$\begin{aligned}
PA^2 + PB^2 + PC^2 &= PA^2 + PB^2 + (PR + RC)^2 \\
&= PA^2 + PB^2 + PR^2 + RC^2 + 2PR.RC \\
&= PA^2 + 2PB^2 + PR^2 + 2PR.PB \\
&\qquad (PB = RC \text{ by symmetry}) \\
&= PA^2 + PR^2 + 2PB(PB + PR) \\
&= QA^2 + 2PB(PB + PR) \quad \text{(by Pythagoras)} \\
&= 2r^2 + 2PB.PC \\
&\qquad (\text{since } \angle APR = 90^\circ \text{ and } RA \\
&\qquad \text{is a diameter of } K) \\
&= 4r^2 + 2(s^2 - r^2) \quad \text{(secant-chord theorem)} \\
&= 2(r^2 + s^2).
\end{aligned}$$

Solution 6 (Colin Lu, Melbourne Grammar School, VIC)

We have the following factorizations of numbers appearing in the problem:

$$\begin{aligned}
845 &= 5 \cdot 13^2 \\
325 &= 5^2 \cdot 13 \\
847 &= 7 \cdot 11^2 \\
539 &= 7^2 \cdot 11 \\
245 &= 5 \cdot 7^2 \\
175 &= 5^2 \cdot 7 \\
1889 &= 11 \cdot 13^2 \\
1573 &= 11^2 \cdot 13.
\end{aligned}$$

Let $a = \sqrt[3]{5}$, $b = \sqrt[3]{7}$, $c = \sqrt[3]{11}$ and $d = \sqrt[3]{13}$. Note that

$$\begin{aligned}
\tfrac{a^3+d^3}{3} &= 6 \\
\tfrac{b^3+c^3}{3} &= 6 \\
\tfrac{a^3+b^3}{3} &= 4 \\
\tfrac{c^3+d^3}{3} &= 8.
\end{aligned}$$

Using this we may compute

$$\begin{aligned}\sqrt[3]{6}+\sqrt[3]{845}+\sqrt[3]{325} &= \sqrt[3]{\tfrac{a^3+d^3}{3}}+ad^2+a^2d\\ &= \tfrac{\sqrt[3]{a^3+d^3+3ad^2+3a^2d}}{\sqrt[3]{3}}\\ &= \tfrac{a+d}{\sqrt[3]{3}}\end{aligned}$$

where we have used the identity $(a+d)^3 = a^3+3a^2d+3ad^2+d^3$.
Similarly, we have

$$\begin{aligned}\sqrt[3]{6}+\sqrt[3]{847}+\sqrt[3]{539} &= \tfrac{b+c}{\sqrt[3]{3}}\\ \sqrt[3]{4}+\sqrt[3]{245}+\sqrt[3]{175} &= \tfrac{a+b}{\sqrt[3]{3}}\\ \sqrt[3]{8}+\sqrt[3]{1889}+\sqrt[3]{1573} &= \tfrac{c+d}{\sqrt[3]{3}}.\end{aligned}$$

Thus the LHS and RHS of the expression given in the problem are both equal to $\frac{a+b+c+d}{\sqrt[3]{3}}$. Hence LHS = RHS, as required.

Note that we have shown more, namely, the common value is equal to

$$\frac{\sqrt[3]{5}+\sqrt[3]{7}+\sqrt[3]{11}+\sqrt[3]{13}}{\sqrt[3]{3}}.$$

Solution 7

Alternative 1

We note that $2010 = 2\times 3\times 5\times 67$ is the product of 4 distinct primes. Let $P = \{2,3,5,67\}$. We say that a list of numbers is *distinguished by* m if no two numbers in the list give the same remainder when divided by m.

Fact 1. If a list is distinguished by m, it is also distinguished by any multiple of m.

Fact 2. If $0 < x < y < 2010$, then $\{x,y\}$ is distinguished by some p in P.

These facts, which are simple corollaries of the Chinese Remainder Theorem, will be used repeatedly in our proof.

By Fact 2, $\{a,b\}$ is distinguished by some p_1 in P. Hence there is at most one x in $\{a,b\}$ for which $\{c,x\}$ is not distinguished by p_1. We can choose p_2 in P so that $\{c,x\}$ is distinguished by p_2 (if x does not exist, choose p_2 in P arbitrarily).

Then $\{a,b,c\}$ is distinguished by $\operatorname{lcm}(p_1,p_2)$.

Now there is at most one y in $\{a,b,c\}$ for which $\{d,y\}$ is not distinguished by $\operatorname{lcm}(p_1,p_2)$. We can choose p_3 in P so that $\{d,y\}$ is distinguished by

p_3 (if y does not exist, choose p_3 in P arbitrarily). Then $\{a, b, c, d\}$ is distinguished by $\text{lcm}(p_1, p_2, p_3)$.

Letting $e = \text{lcm}(p_1, p_2, p_3)$, it is clear that (i), (ii) and (iii) hold.

Alternative 2 (Norman Do)

Construct a graph with four vertices labeled a, b, c, d. We connect the two vertices x and y by an edge if $x - y$ is divisible by at least three of the four primes 2, 3, 5, 67. Furthermore, we label this edge with the primes $x - y$ is divisible by (out of 2, 3, 5, 67).

Note that $x - y$ is never divisible by all four of the primes, so the labels are always triples. If one of the triples does not occur as an edge label, we can take e to be the product of the three primes in that triple and are done. Otherwise, there must be some edge labelled 2, 3, 5, some edge labelled 2, 3, 67, some edge labelled 2, 5, 67 and some edge labelled 3, 5, 67. The subgraph formed by these four edges must form a cycle (of length 3 or 4).

Now we use the fact that if there is a path from x to y, then $x - y$ is divisible by all the primes which appear on all the edges of the path. If there is a 4-cycle, say $abcda$, then $a - d$ is divisible by 2, 3, 5 and 67, a contradiction. Similarly if there is a 3-cycle, say $abca$, then $a - c$ is divisible by 2, 3, 5 and 67, again a contradiction. This proves the claim.

Alternative 3 (Jamie Simpson)

We note that $2010 = 2 \times 3 \times 5 \times 67$. If a counter-example exists then for each member of the set

$$\left\{ \frac{2010}{p} : p \in \{2, 3, 5, 67\} \right\}$$

there exists a pair $\{x, y\}$ from $\{a, b, c, d\}$ such that $\frac{2010}{p}$ divides $\{x, y\}$. The four pairs must be distinct else we have a pair with $x - y$ divisible by $\frac{2010}{p_1}$ and $\frac{2010}{p_2}$ for distinct members p_1 and p_2 of $\{2, 3, 5, 67\}$.

This would mean 2010 divides $x - y$ which is impossible as $0 < |x - y| < 2010$. We can construct a graph with a, b, c, d as vertices and 4 edges corresponding to the 4 pairs.

Such a graph must contain a cycle, either C_3 or C_4. If the cycle is C_3 then suppose, without loss of generality, that its vertices are a, b, c.

Then there exist p_1, p_2 and p_3 from $\{2, 3, 5, 67\}$ such that $\frac{2010}{p_1}$ divides $c - a$, $\frac{2010}{p_2}$ divides $b - a$ and $\frac{2010}{p_3}$ divides $c - b$.

Then $\frac{2010}{p_1 p_2}$ divides $(c - a) - (b - a) = c - b$ which means $\text{lcm}(\frac{2010}{p_1 p_2}, \frac{2010}{p_3})$ divides $c - b$, which is impossible.

Suppose the cycle is C_4 and, without loss of generality, that it is $abcd$.

Then for some ordering p_1, p_2, p_3, p_4 of 2, 3, 5, 67 we have $\frac{2010}{p_1}$ divides $b-a$, $\frac{2010}{p_2}$ divides $c-b$, $\frac{2010}{p_3}$ divides $d-a$ and $\frac{2010}{p_4}$ divides $d-c$. But then $\frac{2010}{p_1 p_2}$ and $\frac{2010}{p_3 p_4}$ both divide $c-a$, which means 2010 divides $c-a$. This is again impossible.

Alternative 4 (Sampson Wong, James Ruse Agricultural High School, NSW)

Let p_1, p_2, p_3, p_4 be the primes $2, 3, 5, 67$. Let $e_i = \frac{2010}{p_i}$ for $i = 1, 2, 3, 4$. We shall prove that at least one of the e_i satisfies the conditions.

Note that each e_i satisfies (i) and (ii). Assume for the sake of contradiction that none of the e_i satisfy (iii). Thus for each $i \in \{1, 2, 3, 4\}$ there exists a pair x, y of distinct elements of $\{a, b, c, d\}$ such that $x \equiv y \pmod{e_i}$.

Define a graph G with four vertices labeled a, b, c, d as follows. Two vertices x, y are joined by an edge with label e_i if $x \equiv y \pmod{e_i}$. Note that G has at least four edges, one for each e_i. It is well known, and easy to prove, that any graph on n vertices which has at least n edges, must contain a cycle. In our case $n = 4$ and so G must contain a cycle.

The definition of G precludes it from having a 1-cycle.

If G contained a 2-cycle, there would exist two distinct members x, y of $\{a, b, c, d\}$ such that $x \equiv y \pmod{e_i}$ and $y \equiv x \pmod{e_j}$ for $i \neq j$. Since $\operatorname{lcm}(e_i, e_j) = 2010$ for $i \neq j$ it follows that $x \equiv y \pmod{2010}$, a contradiction.

If G contained a 3-cycle, there would exist three distinct members x, y, z of $\{a, b, c, d\}$ such that

$$\begin{array}{rcll} x & \equiv & y & \pmod{e_i} \\ y & \equiv & z & \pmod{e_j} \\ z & \equiv & x & \pmod{e_k} \end{array}$$

for some i, j, k distinct members of $\{1, 2, 3, 4\}$. Since $p_k | e_i$ and $p_k | e_j$, the first two congruences imply that $x \equiv z \pmod{p_k}$. This along with the third congruence implies that $x \equiv z \pmod{2010}$ since $\operatorname{lcm}(p_k, e_k) = 2010$. This is a contradiction.

If G contained a 4-cycle, there would exist a permutation (x, y, z, w) of (a, b, c, d) such that

$$\begin{array}{rcll} x & \equiv & y & \pmod{e_i} \\ y & \equiv & z & \pmod{e_j} \\ z & \equiv & w & \pmod{e_k} \\ w & \equiv & z & \pmod{e_\ell} \end{array}$$

where (i, j, k, ℓ) is a permutation of $(1, 2, 3, 4)$. Since $p_k, p_\ell | e_i, e_j$, the first two conguences imply that $x \equiv z \pmod{p_k p_\ell}$. Since $p_i, p_j | e_k, e_\ell$, the last two congruences imply that $x \equiv z \pmod{p_i p_j}$.

Since $\operatorname{lcm}(p_k p_\ell, p_i p_j) = 2010$, we conclude that $x \equiv z \pmod{2010}$, a contradiction.

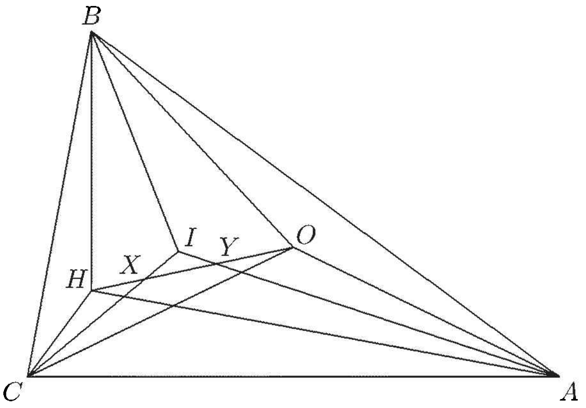
Thus in all cases G contains no cycle. This contradicts our earlier observation about G. Thus one of the e_i is a suitable candidate for e.

Solution 8

Alternative 1 (Aaron Chong, Doncaster Secondary College, VIC)

Let $\alpha = \angle A$, $\beta = \angle B$, $\gamma = \angle C$. Let I be the incentre of triangle ABC. Standard angle chasing yields $\angle CBH = 90^\circ - \gamma$, $\angle CBI = \frac{\beta}{2} = 90^\circ - \frac{\alpha+\gamma}{2}$ and $\angle CBO = 90^\circ - \alpha$. From $\alpha < \beta < \gamma < 90^\circ$ it follows that rays BC, BH, BI, BO, BA lie in that order around B. Similar angle chases show rays CB, CH, CI, CO, CA lie in that order around C and that rays AC, AH, AI, AO, AB lie in that order around A. Specifically, around point C we have $\angle BCH = 90^\circ - \beta$, $\angle BCI = 90^\circ - \frac{\alpha+\beta}{2}$, $\angle BCO = 90^\circ - \alpha$ and around point A we have $\angle CAH = 90^\circ - \gamma$, $\angle CAI = 90^\circ - \frac{\beta+\gamma}{2}$, $\angle CAO = 90^\circ - \beta$.

Furthermore, since H and O are isogonal conjugates in triangle ABC, it follows that BI bisects $\angle HBO$, CI bisects $\angle HCO$ and AI bisects $\angle HAO$. Let CI and AI intersect HO at X and Y, respectively. Hence X and Y both lie on the segment OH. Thus the diagram appears as follows.

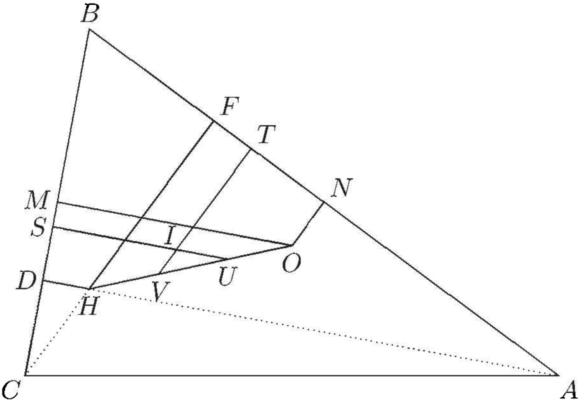


Now $\angle CHO > \angle CHA = 180^\circ - \beta > 90^\circ$. Hence $CO > CH$. We also have $\angle HOA > \angle COA = 2\beta > 90^\circ$. ($2\beta \leq 90^\circ \Rightarrow \alpha + \beta < 90^\circ \Rightarrow \gamma > 90^\circ$). Hence $HA > AO$.

By the angle bisector theorem we have $\frac{HX}{XO} = \frac{CH}{CO} < 1$. Thus $HX < XO$. If follows that X is closer to H than to O. Similarly, $\frac{HY}{YO} = \frac{HA}{AO} > 1$. Thus Y is closer to O than to H. Hence the points H, X, Y, O lie in that order on segment HO. This implies that I, which is the intersection of CX and AY, lies on the same side of HO as B. Since BI is the angle bisector of $\angle HBO$, it follows that I lies inside triangle BHO.

Alternative 2 (Sampson Wong, James Ruse Agricultural High School, NSW)

Let I be the incentre of triangle ABC. Let $a = BC$, $b = AC$, $c = AB$. From $\angle A < \angle B < \angle C < 90^\circ$ we have $a < b < c$. Let M, S, D be the feet of the perpendiculars from O, I, H to BC. Let N, T, F be the feet of the perpendiculars from O, I, H to AB.



We claim that the points B, M, S, D, C occur in that order on the segment BC. We know $BM = \frac{a}{2}$. Also S is the point of contact of the incircle with BC, thus $BS = s - b = \frac{a+c-b}{2}$ from the standard incircle substitution. Finally, $BD = c \cdot \cos \angle B = c \cdot \frac{a^2+c^2-b^2}{2ac} = \frac{a^2+c^2-b^2}{2a}$ from the cosine rule. We have

$$\begin{array}{rcccc} & BM & < & BS & < & BD \\ \Leftrightarrow & \frac{a}{2} & < & \frac{a+c-b}{2} & < & \frac{a^2+c^2-b^2}{2a} \\ \Leftrightarrow & 0 & < & c-b & < & \frac{c^2-b^2}{a} \\ \Leftrightarrow & 0 & < & a(c-b) & < & (c-b)(c+b), \end{array}$$

which is true since $b < c$ and $a < b + c$ from the triangle inequality.

We also claim that points A, N, T, F, B occur in that order on segment AB. Similarly, we have $BN = \frac{c}{2}$, $BT = \frac{a+c-b}{2}$ and $BF = \frac{a^2+c^2-b^2}{2c}$. We have

$$\begin{array}{rccccc} & BF & < & BT & < & BN \\ \Leftrightarrow & \frac{a^2+c^2-b^2}{2c} & < & \frac{a+c-b}{2} & < & \frac{c}{2} \\ \Leftrightarrow & \frac{a^2-b^2}{c} & < & a-b & < & 0 \\ \Leftrightarrow & (a-b)(a+b) & < & c(a-b) & < & 0, \end{array}$$

which is true since $a < b$ and $a + b > c$ from the triangle inequality.

Hence we have established that S is between M and D and that T is between N and F.

Let us now extend SI and TI to intersect OH at U and V, respectively. Note from our previous observation that this implies that U and V both lie on the segment OH. From the parallel lines MO, SU and DH, we have

$$\frac{OU}{UH} = \frac{MS}{SD} = \frac{\frac{a+c-b}{2} - \frac{a}{2}}{\frac{a^2+c^2-b^2}{2a} - \frac{a+c-b}{2}} = \frac{a(c-b)}{(c-b)(c+b-a)} = \frac{a}{c+b-a}$$

and similarly

$$\frac{OV}{VH} = \frac{NT}{TF} = \frac{c}{a+b-c}.$$

Thus

$$\begin{array}{rccc} & \frac{OU}{UH} & < & \frac{OV}{VH} \\ \Leftrightarrow & \frac{a}{c+b-a} & < & \frac{c}{a+b-c} \\ \Leftrightarrow & a(a+b-c) & < & c(c+b-a) \\ \Leftrightarrow & 0 & < & (c-a)(a+b+c). \end{array}$$

Hence since $c > a$ we conclude that U is closer to O than V. Thus the points O, U, V, H occur in that order on segment OH. It follows that the rays SI and TI must have intersected before reaching OH. This means that B and I lie on the same side of OH. However, since O and H are isogonal conjugates and BI is an internal angle bisector, we have that BI lies between BO and BH. Thus I lies inside triangle BOH.

RESULTS

Number of Candidates: 118 (New Zealand participated for the first time)

Gold

CHONG Aaron*	Doncaster Secondary College	VIC	12
WONG Sampson	James Ruse Agricultural High School	NSW	12
VASAK David	Sydney Grammar School	NSW	12
GRANVILLE Malcolm	Auckland Grammar School	NZ	12
LAW Stacey	James Ruse Agricultural High School	NSW	12
De SILVA Raveen	James Ruse Agricultural High School	NSW	12
PARK Kiho	Mount Waverley Secondary College	VIC	12
XU Yanning	St Peter's College	SA	10
LARGE Tim	Sydney Grammar School	NSW	11
LEI Yitao	All Saints Anglican School	QLD	12
LIANG Alfred	Trinity Grammar School	VIC	12
RYBA Christopher	The Hutchins School	TAS	12

Silver

YU Angel	Perth Modern School	WA	11
HOANG Anh Nhat	Morley Senior High School	WA	12
LU Colin	Melbourne Grammar School	VIC	11
ZHANG Si Cong Scott	Auckland International College	NZ	11
FU Nancy	Pymble Ladies' College	NSW	10
WORMELL John	Sydney Boys High School	NSW	12
WANG Yuan Edward	Hillcrest High School	NZ	12
ZHANG Robert	Auckland Grammar School	NZ	10
YAN Tom	Auckland Grammar School	NZ	11
LAM Kevin	James Ruse Agricultural High School	NSW	12
KHOU Victor	James Ruse Agricultural High School	NSW	10
CHO Byung-Cheol	Auckland Grammar School	NZ	9
HO Jonathan	St Joseph's College	QLD	12
GOREY Declan	Sydney Boys High School	NSW	11

LEE Vickie	Pymble Ladies' College	NSW	11
MACKERETH Stephen	King's College	NZ	12
CHEN Yuanjie	St Peter's College	SA	12
WANG Tzu Min Michael	MacLean's College	NZ	11

Australian Team to participate in 2010 IMO at Astana, Kazakhstan

Aaron Chong, Sampson Wong, David Vasak, Stacey Law, Kiho Park, Tim Large.

2011

Problems

PAPER 1 Tuesday 15 February

1. Let b be a fixed integer greater than or equal to 2.

For each positive integer n, let $S_b(n)$ be the sum of the digits of n when n is expressed in base b.

For example, if $b = 4$, then $S_4(26) = S_4\left(1 \times 4^2 + 2 \times 4 + 2 \times 1\right) = 1 + 2 + 2 = 5$.

Determine all positive integers m with the property that

for all n, whenever m is a factor of $S_b(n)$, then m is also a factor of $S_b(n+1) - 1$.

2. The vertices of the regular polygon $P_1P_2 \ldots P_{2n}$ lie on a circle of radius 1.

For $1 \leq i \leq n-1$, let a_i be the length of the line segment P_1P_{i+1}.

Prove that

$$\left(\frac{4}{a_1^2} - 1\right)\left(\frac{4}{a_2^2} - 1\right) \cdots \left(\frac{4}{a_{n-1}^2} - 1\right) = 1.$$

3. Let A, B and C be three distinct points on a circle of radius r.

Prove that triangle ABC has an obtuse angle if and only if there exists a point X in the plane such that the distances AX, BX and CX are all less than r.

4. Determine all functions f defined for non-negative integers and taking non-negative integers as values such that $f(1) > 0$ and

$$(f(x))^2 + (f(y))^2 = f\left(x^2 + y^2\right),$$

for all non-negative integers x and y.

PAPER 2 Wednesday 16 February

5. Let $PQRS$ be a rectangle with centre O.

Let E and F be the midpoints of PQ and QR, respectively.

Let A, B, C, D be points on sides PQ, QR, RS, SP, respectively, such that $ABCD$ is a rhombus and A lies between E and Q.

Let K be the intersection of AB and EF.

Prove that OK is perpendicular to AB.

6. Determine all real numbers r such that the three solutions of the equation

$$x^3 - 30x^2 + rx - 780 = 0$$

are the side lengths of a right-angled triangle.

7. Determine all positive integers n with the following property:

for each integer k there exists an integer i such that $\frac{i(i-1)}{2} - k$ is divisible by n.

8. Let S be the set of all positive integers that are less than or equal to 2211.

Let T be a subset of S containing 2011 elements.

Prove that there is a number in T that is the sum of 11, not necessarily distinct, numbers from T.

Solution 1

Alternative 1 (Declan Gorey, Sydney Boys High School, NSW)

We first note that if the last digit of n is not $b-1$, then $S_b(n+1) = S_b(n)+1$.

1. Suppose $m|b-1$.

   Let A represent the digit $b-1$. Write $n = X\underbrace{AA\ldots A}_{c \text{ digits}}$, where the last digit of X is not A.

   Then

$$\begin{aligned} S_b(n)+1 &= S_b(X)+c(b-1)+1 \\ &\equiv S_b(X)+1 \pmod{m} \\ &= S_b(X+1) \\ &= S_b(n+1). \end{aligned}$$

2. Now suppose that whenever m is a factor of $S_b(n)$, then m is also a factor of $S_b(n+1)-1$.

   Let $n = \underbrace{11\ldots1}_{a \text{ digits}}0A$, where a is chosen so that $a+b-1 = S_b(n) \equiv 0 \pmod{m}$. Then $a+1 = S_b(n+1) \equiv 1 \pmod{m}$. Hence $m|a$, whence $m|b-1$.

Answer: m has the given property if and only if $m|b-1$.

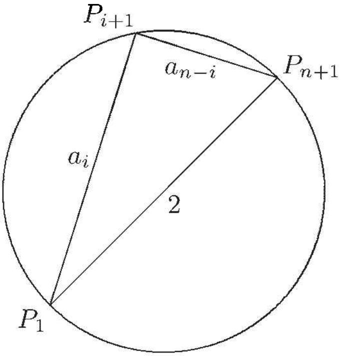
Alternative 2 (Tim Large, Sydney Grammar School, NSW)

It is easy to see that $S_b(n) \equiv n \pmod{b-1}$, using the fact that $b^a \equiv 1 \pmod{b-1}$. For this reason, any divisor of $b-1$ works. Now suppose that m works and consider the number n whose first digit is $b-1$, whose second digit is 0, and whose remaining $m-1$ digits are $b-1$. Clearly, $S_b(n) = m(b-1)$, while $S_b(n+1)-1 = b-1$. Therefore, m must be a divisor of $b-1$.

Solution 2

Alternative 1 (Nancy Fu, James Ruse Agricultural High School, NSW)

As P_1P_{n+1} is a diameter of the circumcircle of the polygon, $\triangle P_1P_{i+1}P_{n+1}$ is right-angled with side lengths 2, a_i and a_{n-i},



so that, by Pythagoras, $a_i^2 + a_{n-i}^2 = 4$, i.e. $4 - a_i^2 = a_{n-i}^2$. Therefore,

$$\begin{aligned}
\prod_{i=1}^{n-1}\left(\frac{4}{a_i^2} - 1\right) &= \prod_{i=1}^{n-1}\left(\frac{4 - a_i^2}{a_i^2}\right) \\
&= \frac{\displaystyle\prod_{i=1}^{n-1} a_{n-i}^2}{\displaystyle\prod_{i=1}^{n-1} a_i^2} \\
&= \prod_{i=1}^{n-1}\frac{a_i^2}{a_i^2} \\
&= 1.
\end{aligned}$$

Alternative 2 (Angel Yu, Perth Modern School, WA)

We use the sine rule:

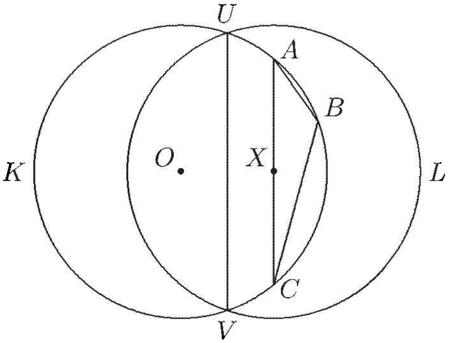
$$\frac{4}{a_i^2} - 1 = \frac{1}{\sin^2 P_i} - 1 = \frac{\cos^2 P_i}{\sin^2 P_i} = \frac{\sin^2(90^\circ - P_i)}{\sin^2 P_i},$$

where $P_i = \angle P_1P_{n+1}P_{i+1}$. Note that $90^\circ - \angle P_i = P_{n-i}$. We may now use the symmetry of the P_i in the same way as in *Alternative 1* to show the product is 1.

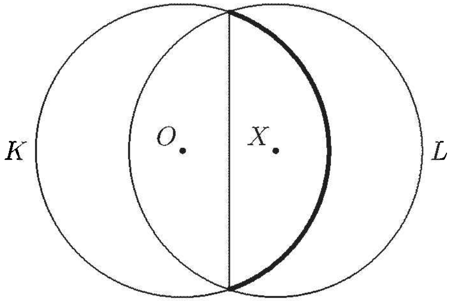
Solution 3

Alternative 1 (Rachel Wong, James Ruse Agricultural High School, NSW)

(i) Let K be the circle on which A, B, C lie. Suppose triangle ABC is obtuse with $\angle ABC > 90°$. Then its circumcentre, O say, lies outside the triangle and on the opposite side of AC to B. Let X be the midpoint of AC. Let L be the circle with centre X and radius r. Let U, V be the intersection points of K and L. Note that UV is parallel to AC and lies in between AC and O. Furthermore, the region bounded by the minor arc UV in K and the minor arc UV in L is precisely the intersection of circles K and L. Thus since A, B and C all lie strictly inside circle L we have $AX, BX, CX < r$.

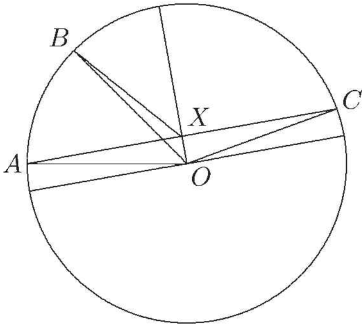


(ii) Let X be any point in the plane. Let K be the circumcircle of ABC and let L be the circle with centre X and radius r. If $AX, BX, CX < r$, then A, B, C must lie on the portion of the arc of circle K that is inside circle L. But no triangle with its vertices on this arc will contain O because the arc is bounded by the common chord of K and L, which lies between O and X. Therefore the triangle has an obtuse angle.



Alternative 2 (Problems Committee)

(i) Let K be the circle on which A, B, C lie. Suppose $\angle ABC > 90°$. Then A, B, C all lie in one and the same of the half planes determined by the line through the centre O of K that is parallel to AC. Let X be the midpoint of AC.



Then $AX < AO = r$, as AO is the hypotenuse of the right-angled triangle OXA.

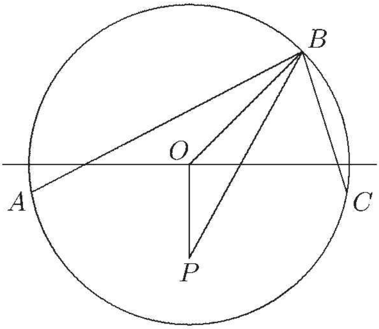
Similarly, $CX < r$. If O, X, B are collinear, then $BX < BO = r$.

If O, X, B are not collinear, then $\angle OXB > 90°$, and $BX < BO = r$ follows.

(ii) Now suppose triangle ABC has no obtuse angle and P is a point in the plane.

If $P = O$, then $AP = BP = CP = r$.

Now let $P \neq O$.



Since triangle ABC has no obtuse angle, not all three points A, B, C can lie in one and the same half plane, determined by a line through O, as P. In particular, P and one of these three points, say B, do not lie in one and the same half plane determined by the diameter of K that is perpendicular to OP.

Therefore $\angle POB \geq 90°$, so that $BP \geq BO = r$.

Solution 4 (Angel Yu, Perth Modern School, WA)

We note that $f(x) = x$ obviously is a solution.

Putting $x = y = 0$, we have $2\left(f(0)\right)^2 = f(0)$. Since $f(0)$ is an integer, it follows that $f(0) = 0$.

Next, putting $x = 1$, $y = 0$, we obtain $\left(f(1)\right)^2 + \left(f(0)\right)^2 = f(1)$, thus $\left(f(1)\right)^2 = f(1)$, so that $f(1) = 1$.

Putting $x = y = 1$ then yields $f(2) = 2$. Putting $x = 2$ and $y = 0$, 1 or 2 gives $f(4) = 4$, $f(5) = 5$ and $f(8) = 8$.

To evaluate $f(3)$, $f(6)$ and $f(7)$, we note that $t^2 + u^2 = v^2 + w^2$ implies

$$\left(f(t)\right)^2 + \left(f(u)\right)^2 = \left(f(v)\right)^2 + \left(f(w)\right)^2 \qquad (1).$$

As $3^2 + 4^2 = 0^2 + 5^2$, we thus have

$$\left(f(3)\right)^2 + \left(f(4)\right)^2 = \left(f(0)\right)^2 + \left(f(5)\right)^2,$$

i.e. $\left(f(3)\right)^2 + 16 = 25$, or $f(3) = 3$.

Putting $x = 3$, $y = 0$, we thus get $f(9) = 9$, and putting $x = 3$, $y = 1$, we thus get $f(10) = 10$.

As $6^2 + 8^2 = 0^2 + 10^2$, we thus have

$$\left(f(6)\right)^2 + \left(f(8)\right)^2 = \left(f(0)\right)^2 + \left(f(10)\right)^2,$$

i.e. $\left(f(6)\right)^2 + 64 = 100$, or $f(6) = 6$.

As $7^2 + 1^2 = 5^2 + 5^2$, we thus have

$$\left(f(7)\right)^2 + \left(f(1)\right)^2 = \left(f(5)\right)^2 + \left(f(5)\right)^2,$$

i.e. $\left(f(7)\right)^2 + 1 = 50$, or $f(7) = 7$.

So far we have shown that $f(n) = n$ if $0 \leq n \leq 10$.

This encourages us to suspect that $f(x) = x$ is the only solution.

We prove this by induction.

Suppose that $n > 10$ and that $f(k) = k$ for all $k < n$.

Case 1. n is odd, say $n = 2m + 1$.

We observe that

$$(2m + 1)^2 + (m - 2)^2 = (2m - 1)^2 + (m + 2)^2.$$

Now apply (1) to obtain

$$(f(2m+1))^2 + (f(m-2))^2 = (f(2m-1))^2 + (f(m+2))^2 .$$

By induction,

$$(f(2m+1))^2 + (m-2)^2 = (2m-1)^2 + (m+2)^2,$$

or $f(2m+1) = 2m+1$.

Case 2. n is even, say $n = 2m+2$.

We observe that

$$(2m+2)^2 + (m-4)^2 = (2m-2)^2 + (m+4)^2.$$

Now apply (1) to obtain

$$(f(2m+2))^2 + (f(m-4))^2 = (f(2m-2))^2 + (f(m+4))^2 .$$

By induction,

$$(f(2m+2))^2 + (m-4)^2 = (2m-2)^2 + (m+4)^2,$$

or $f(2m+2) = 2m+2$.

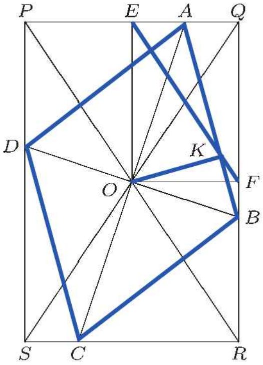
This confirms our suspicion, and so the answer is:

The only function f satisfying the condition of the problem is given by $f(x) = x$.

Solution 5

Alternative 1 (Nancy Fu, James Ruse Agricultural High School, NSW)

Since $ABCD$ is a rhombus, its centre is the midpoint of segments AC and BD. Thus the centre of the rhombus is also the centre of the rectangle $ABCD$ because the midpoint of AC is equidistant from PQ and SR and is also equidistant from PS and QR.



The diagonals of the rhombus are perpendicular. Thus $\angle AOB = 90^\circ$. Since also $\angle EOF = 90^\circ$ we have $\angle EOA = 90^\circ - \angle AOF = \angle FOB$. Hence $\triangle AEO$ and $\triangle BFO$ are similar, so that

$$\frac{OA}{OB} = \frac{OE}{OF},$$

which implies that $\triangle EFO$ and $\triangle ABO$ are also similar. We thus have

$$\angle OEK = \angle OEF = \angle OAB = \angle OAK,$$

from which follows that $OEAK$ is a cyclic quadrilateral.

Therefore

$$\angle AKO = 180^\circ - \angle AEO = 180^\circ - 90^\circ = 90^\circ,$$

q.e.d.

Alternative 2 (Rachel Wong, James Ruse Agricultural High School, NSW)

Since the opposite sides of the rectangle are parallel, as are the opposite sides of the rhombus, we see that triangles ADP and CBR are similar. Furthermore, these two triangles are congruent because $AD = CB$. Therefore if we rotate the diagram by 180° about O, the diagram lands exactly on itself with $A \to C$ and $B \to D$. This implies that O is also the centre of the rhombus. The remainder of the solution is as in *Alternative 1*.

Alternative 3 (Tim Large, Sydney Grammar School, NSW)

As in *Alternative 2*, it is proven that O is also the centre of the rhombus. Note that $AQBO$ is a cyclic quadrilateral due to $\angle AQB = \angle AOB = 90^\circ$. Thus O lies on the circumcircle of triangle AQB. Consider the Simson's line of O with respect to triangle AQB. Since $OE \perp AQ$ and $OF \perp BQ$, this Simson's line is in fact the line EF. Since EF intersects AB at K, it follows from the theorem about Simson's lines that $OK \perp AB$, as required.

Alternative 4 (Ivan Guo)

Once it is shown that O is the centre of the rhombus we proceed as follows. Being a rectangle, $OEQF$ is a cyclic quadrilateral. Also $AOBQ$ is a cyclic quadrilateral (opposite right angles). Apply the pivot theorem about triangle BFK, we get $AEOK$ is cyclic.

So $\angle AKO = 180^\circ - \angle AEO = 180^\circ - 90^\circ = 90^\circ$, as required.

Solution 6 (Declan Gorey, Sydney Boys High School, NSW)

Let a, b, c be the lengths of such a triangle. Without loss of generality, we may assume that $0 < a \leq b < c$. Then

$$\begin{aligned} x^3 - 30x^2 + rx - 780 &= (x-a)(x-b)(x-c) \\ &= x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc, \end{aligned}$$

so that

$$\begin{aligned} 30 &= a+b+c, && (1) \\ r &= ab+ac+bc, && (2) \\ 780 &= abc. && (3) \end{aligned}$$

By (1), we have

$$(a+b)^2 = (30-c)^2,$$

while, by Pythagoras,

$$c^2 = a^2 + b^2,$$

so that

$$0 = 900 - 60c - 2ab.$$

Hence, since $ab = \frac{780}{c}$ by (3), this simplifies to

$$c^2 - 15c + 26 = 0,$$

implying that either $c = 2$ or $c = 13$.

If $c = 2$, then $0 < a$, $b < 2$ and $a + b + c \neq 30$.

If, however, $c = 13$, then, by (3), $ab = 60$ while, by (1), $a+b = 17$ so that a, b are roots of the quadratic $x^2 - 17x + 60 = 0$. Hence $(a, b) = (5, 12)$ or $(12, 5)$. Therefore, by (2), $r = ab + ac + bc = 60 + 65 + 156 = 281$.

Conversely, one verifies immediately, that 5, 12, 13 are the solutions of the equation $x^3 - 30x^2 + 281x - 780 = 0$.

Answer: $r = 281$.

Solution 7

Alternative 1 (Colin Lu, Melbourne Grammar School, VIC)

The question is equivalent to finding all n such that $\dfrac{i(i-1)}{2}$ takes every value mod n.

Suppose that an odd prime p divides n. Then $\dfrac{i(i-1)}{2}$ takes every value mod p. If we set $a_k = \dfrac{k(k-1)}{2}$, then the sequence $a_1, a_2, \ldots$ is periodic mod p with period p. Hence $a_1, a_2, \ldots, a_p$ must be distinct mod p. But this is simply not true, because $a_1 \equiv 0 \equiv a_p \bmod p$.

Thus we are left only with the case when n is a power of 2. We want to show that $\frac{i(i-1)}{2}$ takes all values modulo 2^k for $k = 0, 1, 2 \ldots$. This is the same as showing that $i(i-1)$ takes all even values modulo 2^{k+1}. We claim that as i goes from 1 to 2^k, all these even values are taken exactly once. Suppose to the contrary that

$$a(a-1) \equiv b(b-1) \pmod{2^{k+1}}$$

for some a, b where $1 \leq a < b \leq 2^k$. This may be rearranged as

$$(a+b-1)(a-b) \equiv 0 \pmod{2^{k+1}}.$$

Since $a+b-1$ and $a-b$ are of opposite parity, one of these expressions is odd while the other is divisible by 2^{k+1}. But since $0 < a+b-1 < 2^{k+1}$ and $0 < b-a < 2^k$, neither of these can be divisible by 2^{k+1}. This contradiction establishes the claim and concludes the proof of the problem.

Answer: n is a power of 2.

Alternative 2 (Angelo Di Pasquale)

We want to find all positive integers n such that

$$\left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\}$$

contains a complete set of residues (modulo n).

We claim that n must be a power of 2. For brevity in what follows we define a set S of integers to be complete (mod n) if S contains a complete set of residues (mod n).

1. Suppose that n has an odd prime factor p. Then

$$\begin{aligned}
& \left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\} \quad \text{is complete (mod } n) \\
\Rightarrow\ & \left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\} \quad \text{is complete (mod } p) \\
\Rightarrow\ & \{i(i-1) : i = 1, 2, \ldots\} \\
& \qquad\qquad \text{is complete (mod } p) \text{ (since } p \text{ is odd)} \\
\Rightarrow\ & \left\{\left(i + \frac{p-1}{2}\right)^2 - \left(\frac{p-1}{2}\right)^2 : i = 1, 2, \ldots\right\} \\
& \qquad\qquad \text{is complete (mod } p) \\
\Rightarrow\ & \left\{\left(i + \frac{p-1}{2}\right)^2 : i = 1, 2, \ldots\right\} \quad \text{is complete (mod } p).
\end{aligned}$$

This, however, is false since it is easily shown that there are only $\frac{p+1}{2}$ quadratic residues, while $p > \frac{p+1}{2}$.

2. We prove, by induction, that

$$\left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\}$$

is complete (mod 2^r) for $r = 1, 2, \ldots$.

By direct checking, we can verify this for $r = 0, 1, 2$.

Suppose that

$$\left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\}$$

is complete (mod 2^r) for $r = t$.

Consider now a particular value of k for $r = t + 1$.

Since, by assumption,

$$\left\{\frac{i(i-1)}{2} : i = 1, 2, \ldots\right\}$$

is complete (mod 2^t), there is a value of i such that

$$\frac{i(i-1)}{2} \equiv k \pmod{2^t}.$$

This implies that $\frac{i(i-1)}{2} \equiv k$ or $k + 2^t \pmod{2^{t+1}}$.

If $\frac{i(i-1)}{2} \equiv k \pmod{2^{t+1}}$, we are done.

If $\frac{i(i-1)}{2} \equiv k + 2^t \pmod{2^{t+1}}$, then we have

$$\begin{aligned}\frac{\left(i + 2^{t+1}\right)\left(i + 2^{t+1} - 1\right)}{2} &= \frac{i(i-1)}{2} + 2^{2t+1} + 2^{t+1}i - 2^t \\ &\equiv k \pmod{2^{t+1}},\end{aligned}$$

and we are also done.

Answer: n has the required property if and only if n is a power of 2.

Solution 8

Alternative 1 (Angel Yu, Perth Modern School, WA)

Consider the following 201 disjoint sets which all have the property that one element is equal to the sum of 11, not necessarily distinct, elements of the same set:

$(1,\ 222,\ 2000 = 222 \times 9 + 1 \times 2)$
$(2,\ 219,\ 2192 = 219 \times 10 + 2)$
$(3,\ 218,\ 2183 = 218 \times 10 + 3)$
$(4,\ 217,\ 2174 = 217 \times 10 + 4)$
$(5,\ 215,\ 2155 = 215 \times 10 + 5)$
$(6,\ 216,\ 2166 = 216 \times 10 + 6)$
$(7,\ 214,\ 2147 = 214 \times 10 + 7)$
$(8,\ 213,\ 2138 = 213 \times 10 + 8)$
$(9,\ 212,\ 2129 = 212 \times 10 + 9)$
$(10,\ 211,\ 2120 = 211 \times 10 + 10)$
$(11,\ 210,\ 2111 = 210 \times 10 + 11)$
$(12,\ 208,\ 2092 = 208 \times 10 + 12)$
$(13,\ 207,\ 2083 = 207 \times 10 + 13)$
$(14,\ 206,\ 2074 = 206 \times 10 + 14)$
$(15,\ 205,\ 2065 = 205 \times 10 + 15)$
$(16,\ 204,\ 2056 = 204 \times 10 + 16)$
$(17,\ 203,\ 2047 = 203 \times 10 + 17)$
$(18,\ 202,\ 2038 = 202 \times 10 + 18)$
$(i,\ 11i)$ for $i = 19, 20, \ldots, 201$
$(18,\ 221,\ 401 = 18 \times 10 + 221)$

If at least one element from each of the above 201 sets were missing from T, then T would have at most $2211 - 201 = 2010$ elements, a contradiction. Thus at least one of those sets must be a subset of T. Thus T has the required property.

Alternative 2 (Ivan Guo)

Consider the following 201 disjoint sets:

(201, 2211)
(200, 2200)
...
(20, 220)
(19, 209)
$(18,\ 221,\ 401 = 18 \times 10 + 221)$
$(17,\ 219,\ 2207 = 219 \times 10 + 17)$
$(16,\ 218,\ 2196 = 218 \times 10 + 16)$
...
(8, 210, 2108)
(7, 208, 2087)

...

(1, 202, 2021)

At least one of those sets must be in T.

Alternative 3 (Declan Gorey, Sydney Boys High School, NSW)

Let k be the smallest element of T.

If $k \geq 201$, the $T = \{201, 202, \ldots, 2211\}$ and $11 \times 201 = 2211$.

If $20 \leq k \leq 200$, consider the following $212 - k$ sets

$$\{i, i + 10k\} \text{ for } i = k, k + 1, \ldots, 211.$$

They are disjoint because $211 < 11k$ and they are subsets of S because $211 + 10k \leq 2211$. If each of these sets contained an element not in T, then this would yield $212 - k$ elements greater than or equal to k which are not in T. However, since k is the smallest element of T, there are $k - 1$ numbers below k which are also not in T. Hence there are a total of 211 numbers which are not in T. This contradicts that $S \setminus T$ has 200 elements. Thus at least one of these sets is entirely in T. Thus $i + 10k = i + 10 \times k$ can be entirely fulfilled within T.

If $1 \leq k \leq 19$, divide the $2011 - k$ elements of S which are at least k into blocks of $20k$ consecutive elements. Each block has the form $(a+1, a+2, \ldots, a+20k)$. If there is no desired 11-tuple in T, then since $a+i+10k = (a+i)+10 \times k$, at most one member of the pair $(a+i, a+i+10k)$ can be in T for each i. Thus each block of $20k$ eliminates at least $10k$ members from T. Hence we eliminate at least $f(k) = \left\lfloor \frac{2011-k}{20k} \right\rfloor \times 10k$ elements from being in T. But $f(k) > \left(\frac{2011-k}{20k} - 1\right) \times 10k = \frac{1}{2}(2011 - 21k) \geq \frac{1}{2}(2011 - 21 \times 19) = 806$. Thus T would have far too few elements, a contradiction.

Alternative 4 (Tim Large, Sydney Grammar School, NSW)

Suppose to the contrary that there exists such a set T such that no element of T is the sum of 11 elements of T. Let m be the maximal element of T. Note that since $|T| = 2011$, we have $m \geq 2011$. Thus there are $m - 2011$ elements of $\{1, 2, \ldots, m\}$ which are in $S \setminus T \ldots$(*).

For $i = 1, 2, \ldots, \left\lfloor \frac{m}{11} \right\rfloor$, let $H_i = \{i, m - 10i\}$. N.B. if $i = \frac{m}{11}$ then we set $H_{\frac{m}{11}} = \{\frac{m}{11}\}$. The H_i are disjoint because $1 < 2 < \ldots < \left\lfloor \frac{m}{11} \right\rfloor \leq m - 10\left\lfloor \frac{m}{11} \right\rfloor < m - 10\left(\left\lfloor \frac{m}{11} \right\rfloor - 1\right) < \ldots < m - 10$. Furthermore, if $H_i \subseteq T$ for some i, then since $10 \times i + (m - 10i) = m$ we have a contradiction to the assumption that m is not a sum of 11 elements of T.

Thus each H_i contains an element not in T. This gives a total of at least $\left\lfloor \frac{m}{11} \right\rfloor$ elements in $S \setminus T$ each of which is at most $m - 10$. By our earlier observation (*), we deduce that $m - 2011 \geq \left\lfloor \frac{m}{11} \right\rfloor > \frac{m}{11} - 1$. This implies that $m > 2211$, a contradiction.

Alternative 5 (Problems Committee)

Of the numbers in T let t be the smallest. Note that $t \leq 201$.

For each i with $0 \leq i \leq 2211 - 11t$, consider the sum of the following 11 elements of T all of which are greater than or equal to t:

$$t, t, \ldots, t(10 \text{ times}), t + i.$$

These sums, $11t + i$, are of course in S.

It thus suffices to prove that there is at least one i with $0 \leq i \leq 2211-11t$, for which both $t + i$ and $11t + i$ are in T.

There are exactly $2211 - (t - 1) - 2011 = 201 - t$ numbers in S that are greater than t and are not in T. Hence there are at most $201 - t$ values of i for which $t + i$ is not in T.

Likewise, there are at most $201 - t$ values of i for which $11t + i$ is not in T. So, there are at most $2(201-t) = 402-2t$ values of i for which either $t + i$ or $11t + i$ (or both) is not in T. It follows that there are at least

$$(2211 - 11t + 1) - (402 - 2t) = 1810 - 9t$$

values of i for which both $t + i$ and $11t + i$ are in T. Since $t \leq 201$, we have $1810 - 9t \geq 1$, q.e.d.

Alternative 6 (Ian Wanless)

Suppose the result is not true.

Let $T \cap \{1, 2, \ldots, 201\} = \{x_1, \ldots, x_n\}$ with $x_1 < x_2 < \cdots < x_n$.

Note that n is at least 1, by the pigeonhole principle.

Case 1: $x_1 \leq 50$.

For any integer m, we have $|\{m+1, m+2, \ldots, m+20x_1\} \cap T| \leq 10x_1$ since we cannot have both b and $10x_1+b$ in T, for $b = m+1, m+2, \ldots, m+10x_1$.

Applying this observation with $m = 0, 20x_1, 40x_1, \ldots$ we see that

$$|S \setminus T| > \frac{2211 - 20x_1}{2} > 500,$$

which is a contradiction.

Case 2: $x_1 > 50$.

The set $\{ax_i + (11 - a)x_{i+1} : a = 1, \ldots, 11, i = 1, \ldots, n - 1\} \cup \{11x_n\}$ is a subset of $(S \setminus T) \cap \{202, \ldots, 2211\}$ and consists of $11n - 10$ distinct integers.

But that means $|T| \leq n + 2010 - (11n - 10) = 2020 - 10n \leq 2010$, again a contradiction.

As neither case is possible, the result is proved.

RESULTS

Number of Candidates: 118

Gold

YU Angel	Perth Modern School	WA	12
LARGE Tim	Sydney Grammar School	NSW	12
GOREY Declan	Sydney Boys High School	NSW	12
LU Colin	Melbourne Grammar School	VIC	12
FU Nancy	James Ruse Agricultural High School	NSW	11
XU Yanning	St Peter's College	SA	11
YAN Tom	Auckland Grammar School	NZ	12
CHUA Alexander	Christ Church Grammar School	WA	9
TAN Xin Zheng	All Saints College	WA	12
ZHANG Robert	Auckland Grammar School	NZ	12
AGISILAOU Adrian	Balwyn High School	VIC	11
YANG Andrew	Rossmoyne Senior High School	WA	11

Silver

YOO Edward	All Saints College	WA	10
ALLEN James	Kristin School	NZ	11
HE Patrick	Scotch College	VIC	10
HETTIGE Thushan	Scotch College	VIC	12
WEI Zack	All Saints Anglican School	QLD	12
CHO Byung-Cheol	Auckland Grammar School	NZ	10
KHOU Victor	James Ruse Agricultural High School	NSW	11
MORRISSEY Benedict	Garin College	NZ	12
WONG Rachel	James Ruse Agricultural High School	NSW	10
ZHANG Allan	James Ruse Agricultural High School	NSW	12
HA Jason	Scotch College	VIC	11

HAN George	Westlake Boys High School	NZ	8
JANG Yoon-Seok	Prince Alfred College	SA	12
PAPANTONIOU John	Sydney Grammar School	NSW	11
CHAPMAN Kaimyn	All Saints Anglican School	QLD	11
LEE Vickie	Pymble Ladies' College	NSW	12
SHU Mel	Melbourne Grammar School	VIC	9
WONG Aaron	Aquinas College	WA	12

Australian Team to participate in 2011 IMO at Amsterdam, Netherlands

Nancy Fu, Declan Gorey, Tim Large, Colin Lu, Yanning Xu, Angel Yu.

AMOC HONOUR ROLL 1979 to 2011

Because of changing titles and affiliations, the most senior title achieved and latest affiliations are generally used, except for the Interim committee, where they are listed as they were at the time.

Interim Committee 1979–1980

Mr P J O'Halloran, Canberra College of Advanced Education, Chair
Prof A L Blakers, University of Western Australia
Dr J M Gani, Australian Mathematical Society
Prof B H Neumann, Australian National University
Prof G E Wall, University of Sydney
Mr J L Williams, University of Sydney

Australian Mathematical Olympiad Committee

The Australian Mathematical Olympiad Committee was founded at a meeting of the Australian Academy of Science at its meeting of 2–3 April 1980.

* denotes Executive Position

Chair*

Prof B H Neumann, Australian National University, 7 years; 1980–1986
Prof G B Preston, Monash University, 10 years; 1986–1995
Prof A P Street, University of Queensland, 6 years; 1996–2001
Prof C Praeger, University of Western Australia, 10 years; 2002–2011

Deputy Chair*

Prof P J O'Halloran, University of Canberra, 15 years; 1980–1994
Prof A P Street, University of Queensland, 1 year; 1995
Prof C Praeger, University of Western Australia 6 years; 1996-2001
Assoc Prof D Hunt, University of New South Wales, 10 years; 2002-2011

Executive Director*

Prof P J O'Halloran, University of Canberra, 15 years; 1980-1994
Prof P J Taylor, University of Canberra, 17 years; 1994-2011

Secretary*

Prof J C Burns, Australian Defence Force Academy, 9 years; 1980-1988
Vacant, 4 years; 1989-1992
Mrs K Doolan, Victorian Chamber of Mines, 6 years; 1993-1998

Treasurer*

Prof J C Burns, Australian Defence Force Academy, 8 years; 1981-1988
Prof P J O'Halloran, University of Canberra, 2 years; 1989-1990
Ms J Downes, CPA, 5 years; 1991-1995
Dr P Edwards, Monash University, 8 years; 1995-2002
Prof M Newman, Australian National University, 6 years; 2003-2008
Dr P Swedosh, St Leonard's College, Victoria, 3 years; 2009-2011

Director of Mathematics Challenge for Young Australians*

Mr J B Henry, Deakin University, 17 years; 1990-2006
Dr K McAvaney, Deakin University, 5 years; 2006-2011

Director of Training*

Mr J L Williams, University of Sydney, 7 years; 1980-1986
Mr G Ball, University of Sydney, 3 years; 1987-1989
Dr D Paget, University of Tasmania, 6 years; 1990-1995
Dr M Evans, Scotch College, Victoria, 3 months; 1995
Assoc Prof D Hunt, University of New South Wales, 5 years; 1996-2000
Dr A Di Pasquale, University of Melbourne, 11 years; 2001-2011

Chair, Senior Problems Committee

Prof B C Rennie, James Cook University, 1 year; 1980
Mr J L Williams, University of Sydney, 6 years; 1981-1986
Assoc Prof H Lausch, Monash University, 25 years; 1987-2011

Team Leader

Mr J L Williams, University of Sydney, 5 years; 1981-1985
Assoc Prof D Hunt, University of New South Wales, 9 years; 1986, 1989–1990, 1996-2001
Dr E Strzelecki, Monash University, 2 years; 1987–1988
Dr D Paget, University of Tasmania, 5 years; 1991-1995
Dr A Di Pasquale, University of Melbourne, 9 years; 2002-2010
Mr I Guo, University of Sydney, 1 year, 2011

Deputy Team Leader

Prof G Szekeres, University of New South Wales, 2 years; 1981-1982
Mr G Ball, University of Sydney, 7 years; 1983-1989
Dr D Paget, University of Tasmania, 1 year; 1990
Mr J Graham, University of Sydney, 3 years; 1991-1993
Dr M Evans, Scotch College, Melbourne, 3 years; 1994-1996
Dr A Di Pasquale, University of Melbourne, 5 years; 1997-2001
Mr D Mathews, University of Melbourne, 3 years; 2002-2004
Mr N Do, University of Melbourne, 4 years; 2005-2008
Mr I Guo, University of New South Wales, 2 years; 2009-2010
Mr Graham White, University of Sydney, 1 year; 2011

State Directors

Australian Capital Territory

Prof M Newman, Australian National University, 1 year; 1980
Mr D Thorpe, ACT Department of Education, 2 years; 1981-1982
Dr R A Bryce, Australian National University, 7 years; 1983-1989
Mr R Welsh, Canberra Grammar School, 1 year; 1990
Mrs J Kain, Canberra Grammar School, 5 years; 1991-1995
Mr J Carty, ACT Department of Education, 17 years; 1995-2011

New South Wales

Dr M Hirschhorn, University of New South Wales, 1 year; 1980
Mr G Ball, University of Sydney, 16 years; 1981-1996
Dr W Palmer, University of Sydney, 15 years; 1997-2011

Queensland

Dr N H Williams, University of Queensland, 21 years; 1980-2000
Dr G Carter, Queensland University of Technology, 10 years; 2001-2010
Dr V Scharaschkin, University of Queensland, 1 year; 2011

South Australia/Northern Territory

Mr K Hamann, SA Department of Education, 18 years; 1980-1982, 1991-2005
Mr V Treilibs, SA Department of Education, 8 years; 1983-1990
Dr M Peake, Adelaide, 6 years; 2006-2011

Tasmania

Mr J Kelly, Tasmanian Department of Education, 8 years; 1980-1987
Dr D Paget, University of Tasmania, 8 years; 1988-1995
Mr W Evers, St Michael's Collegiate School, 9 years; 1995-2003
Dr K Dharmadasa, University of Tasmania, 8 years; 2004-2011

Victoria

Dr D Holton, University of Melbourne, 3 years; 1980-1982
Mr B Harridge, Melbourne High School, 1 year; 1982
Ms J Downes, CPA, 6 years; 1983-1988
Mr L Doolan, Melbourne Grammar School, 9 years; 1989-1998
Dr P Swedosh, St Leonard's College, 14 years; 1998-2011

Western Australia

Dr N Hoffman, WA Department of Education, 3 years; 1980-1982
Assoc Prof P Schultz, University of Western Australia, 14 years; 1983-1988, 1991-1994, 1996-1999
Assoc Prof W Bloom, Murdoch University, 2 years; 1989-1990
Dr E Stoyanova, WA Department of Education, 7 years; 1995, 2000-2005
Dr G Gamble, University of Western Australia, 6 years; 2006-2011

Editor

Prof P J O'Halloran, University of Canberra, 1 year; 1983
Dr A W Plank, University of Southern Queensland, 11 years; 1984-1994
Dr A Storozhev, Australian Mathematics Trust, 15 years; 1994-2008

Editorial Consultant

Dr O Yevdokimov, University of Southern Queensland, 3 years; 2009-2011

Other Memberships of AMOC (showing organisations represented where applicable)

Mr W J Atkins, Australian Mathematics Foundation, 17 years; 1995-2011
Dr S Britton, University of Sydney, 8 years; 1990-1998
Prof G Brown, Australian Academy of Science, 10 years; 1980, 1986-1994
Dr R A Bryce, 28 years total, ACT Director, 7 years; 1983–1989, Australian Mathematical Society, 8 years; 1991-1998, Mathematics Challenge for Young Australians, 13 years; 1999-2011
Mr G Cristofani, Department of Education and Training, 2 years; 1993-1994
Ms L Davis, IBM Australia, 4 years; 1991-1994
Dr W Franzsen, Australian Catholic University, 9 years; 1990-1998
Dr J Gani, Australian Mathematical Society, 1 year; 1980
Assoc Prof T Gagen, National Mathematics Summer School, 6 years; 1993-1998
Ms P Gould, Department of Education and Training, 2 years; 1995-1996
Prof G M Kelly, University of Sydney, 6 years; 1982-1987
Prof R B Mitchell, University of Canberra, 5 years; 1991-1995
Ms Anna Nakos, Mathematics Challenge for Young Australians, 9 years; 2003-2011
Mr S Neal, Department of Education and Training, 4 years; 1990-1993
Prof M Newman, 27 years total, ACT Director, 1 year; 1980, Australian National University, 13 years; 1986-1998, Mathematics Challenge for Young Australians, 7 years; 1999-2002 and 2009–2011, Treasurer, 6 years, 2003-2008
Prof R B Potts, University of Adelaide, 1 year; 1980
Mr H Reeves, Australian Association of Mathematics Teachers, 11 years; 1988-1998
Mr N Reid, IBM Australia, 3 years; 1988-1990
Mr R Smith, Telecom Australia, 5 years; 1990-1994
Prof P J Taylor, 22 years total, Australian Mathematics Foundation, 5 years; 1990-1994, 17 years Executive Director 1994–2011
Prof N S Trudinger, Australian Mathematical Society, 3 years; 1986-1988

Assoc Prof I F Vivian, University of Canberra, 1 year; 1990
Dr M W White, IBM Australia, 9 years; 1980–1988

Many other mathematicians have served AMOC as problems committee members, school directors, members of problems committees and moderators for the Mathematics Challenge for Young Australians and in other roles. Their names are acknowledged in the annual AMOC book, *Mathematics Contests, The Australian Scene.*

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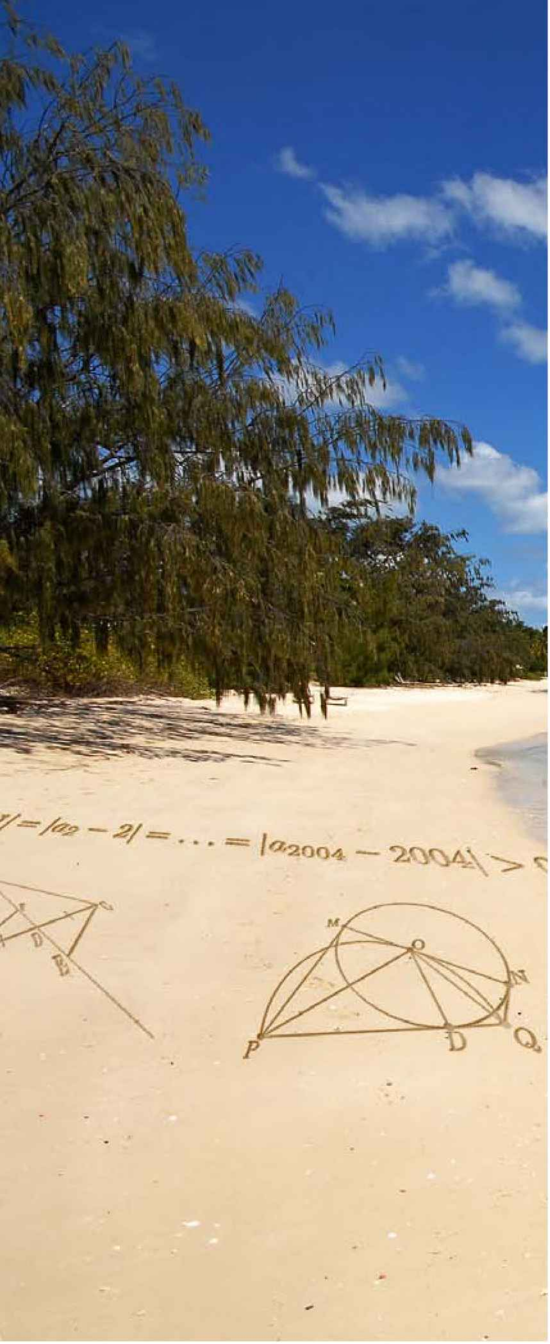

Hans Lausch is a mathematician at Melbourne's Monash University. He has been Chairman of the Australian Mathematical Olympiad Committee's Senior Problems Committee for 23 years and Australia's representative on the Asian Pacific Mathematics Olympiad since its inception. He has composed problems for the International Mathematical Olympiad and his research interests are in algebra and the history of mathematics.

Angelo Di Pasquale, of the University of Melbourne, is Director of Training for the Australian Mathematical Olympiad Committee. He is the Australian Team Leader at the International Mathematical Olympiad. He composes problems for the Australian, Asian Pacific and International Mathematical Olympiads.

David Hunt is a mathematician at the University of New South Wales. He is Deputy Chairman of the Australian Mathematical Olympiad Committee and was its Director of Training for five years and the Australian Team Leader at the International Mathematical Olympiad on nine occasions. He is a joint author of twelve mathematics textbooks published by the Australian Mathematical Sciences Institute.

Peter Taylor is Executive Director of the Australian Mathematics Trust and was Foundation Chairman of the Education Advisory Committee of the Australian Mathematical Sciences Institute. He is a past President of World Federation of National Mathematics Competitions and was co Chair of ICMI Study 16 *Challenging Mathematics in and beyond the Classroom.*

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